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Mass Conservation and the Continuity Equation

The range of applications of fluid mechanics in the natural world is vast. In particular, fluid mechanics shows up in atmospheric physics, geophysics, astrophysics, condensed matter physics, and biological physics, in addition to various engineering disciplines. As amazing as the range of applications is in fluid mechanics, most of the principles involved are the basic principles learned in any first-year physics course. However, when these principles are applied to a moving fluid, the result will be a very complicated set of equations that govern the physics of fluid motion. These equations are known as the **governing equations of fluid motion**, with the famous **Navier–Stokes equations** being front and center. This book will guide you through the details of these equations.

1.1 Conservation in Fluid Mechanics

The equations of fluid mechanics are based, in large part, on conservation principles, particularly the conservation of mass, momentum (which, as we shall see, is a consequence of Newton's laws of motion), and energy. The Navier–Stokes equations come directly from applying momentum conservation (typically in the form of Newton's second law of motion, that is, think *force* = *mass* times *acceleration*[1]) to a moving fluid. Although the Navier–Stokes equations are the most important of the governing equations, they are rarely (if ever) seen in isolation. Instead, they are accompanied by a conservation of mass equation (called the continuity equation or the mass continuity equation) and oftentimes alongside a conservation of energy equation (called the energy equation). The continuity equation is obtained by applying mass conservation to a moving fluid

[1] Note, when we formally discuss the Navier–Stokes equations, we will end up using a more general form of Newton's second law; namely that force is equal to the time derivative of momentum.

Table 1.1 *Conservation principles of fluid mechanics resulting in the governing equations*

Conservation principle[a]	Formal name of principle	Name of resulting governing fluid equation
Mass conservation	Mass conservation	Continuity equation[b]
Momentum conservation	Newton's laws of motion	Navier–Stokes equations[c]
Energy conservation	First law of thermodynamics	Energy equation

[a] Other laws, such as conservation of angular momentum and the second law of thermodynamics (which is not a conservation law), are also very important in fluid mechanics and, as we will see, are "built-in" to the three main equations.
[b] The continuity equation is sometimes used more generally as an equation describing the transport (i.e., movement) of a conserved quantity. In fluid mechanics, however, it is almost always used to describe mass conservation.
[c] Also sometimes called the momentum equations in fluid mechanics.

and the energy equation is obtained by applying energy conservation (in the form of the first law of thermodynamics) to a moving fluid. Table 1.1 summarizes the conservation principles and the name of the corresponding governing equation when each principle is applied to a moving fluid. Incidentally, historically the Navier–Stokes equations refer to the conservation of momentum when applied to a fluid (in particular, as we will discuss, a Newtonian fluid). However, some literature may refer to the whole set of equations tabulated in Table 1.1 as the Navier–Stokes equations. In this book, we will still only refer to momentum conservation as being the Navier–Stokes equations. We are not going to introduce what the equations look like up front as that could be somewhat intimidating. However, if you would like to get a sneak peek as to what the equations look like, you can check out Section 5.4.

Other principles are also important in fluid mechanics in addition to the ones mentioned above, namely the second law of thermodynamics and the conservation of angular momentum. However, these two principles are not always included as separate equations. Instead, as we shall see, they are typically "built-in" to the Navier–Stokes equations and energy equation. Also, note there is an underlying assumption with the governing equations that the fluid of interest is considered to be a continuous medium. In other words, the atomic nature of matter is not considered in the equations and, as a result, the equations start to have trouble making predictions once individual atomic interactions become an important factor. This places fluid mechanics under the umbrella of the more general subject of continuum mechanics. To give a more concrete idea of where such atomic interactions might become important, results using the governing equations to solve for airflow at standard temperature and pressure conditions in channels less than one micron in diameter (which is about a hundred times smaller than the thickness of human hair) might start to deviate from the results obtained in experiments because individual atomic interactions begin to play a role. It should still be stated, however, that even though individual atomic interactions are not accounted for in the governing equations, there are aspects of the equations that can be thought of as modeling atomic interactions in an averaged sense (which we will discuss when we reach the topic of diffusion).

Considering that the Navier–Stokes equations and the other governing equations of fluid motion rely so heavily on conservation principles, we should discuss what we mean by a conservation principle. The main idea behind a conservation principle can be stated, in words, as follows:

the amount of a quantity contained in a system	=	amount of that quantity transferred to the system	−	amount of that quantity transferred out of the system	+	source (+) or sink (−) inside the system

The word "system" is a term adopted from the subject of thermodynamics. **System** simply means the body of interest. The quantity in question can be anything that is conserved; examples include mass, energy, momentum, angular momentum, and electric charge.

In fluid mechanics, we typically think of conservation laws on a per time basis (or time rate basis). Thus, re-stating the idea of a conservation law on a time rate basis would look like the following:

the change of a quantity in a system in a given unit of time	=	time rate of that quantity entering the system	−	time rate of that quantity leaving the system	+	time rate of a source or sink inside the system

A very simple example that may help you understand the conservation principle is to consider the mass of water going in and out of a bathtub. If the bathtub is our system and the quantity of interest is mass, our conservation principle for mass in the bathtub will look like this:

the change of mass of water in the bathtub in a given unit of time	=	time rate of mass of water entering the bathtub	−	time rate of mass of water leaving the bathtub

Notice how we did not bother with any source (or sink) of mass in the bathtub since water does not just appear (or disappear) from within the tub. We can now put some numbers to these terms. Suppose you are filling the bathtub with water from a faucet and the mass is entering the tub at a rate of 0.25 kilograms per second (kg/s). Now suppose you forgot to plug the drain, and the tub is losing water at 0.1 kg/s. The increase in the mass of water in the bathtub per second can be calculated as:

$$\begin{array}{c}\text{the change}\\ \text{of mass of water}\\ \text{in the bathtub}\\ \text{in a given}\\ \text{unit of time}\end{array} = 0.25\text{ kg/s} - 0.1\text{ kg/s} = 0.15\text{ kg/s}$$

If the bathtub can hold 150 kg of water, then it will take about $\frac{150\text{ kg}}{0.15\text{ kg/s}} =$ 1000 s, roughly 17 minutes to fill up.

The rest of the chapter will continue to focus on mass conservation. This will result in the development of the continuity equation.

1.2 Conservation of Mass in One Dimension

To start our more formal discussion of mass conservation, consider the schematic in Figure 1.1. Figure 1.1 illustrates a flow going from left to right (in the x-direction). The flow could be from anything at this point: river, ocean, blood, bathtub, water in a hose, and so forth. Suppose we want to come up with an expression for mass conservation for this flow. To come up with such an expression, let us just focus on a little segment of the flow and do some "accounting" principles for mass. The little segment we are going to focus on is the box-shaped system in the middle of the flow, as shown in Figure 1.1. This box system is assumed to be a region fixed in space. In fluid mechanics, systems that are regions of a flow (often fixed in space) that allow fluid to enter and leave are sometimes given the name **control volume**. As shown in Figure 1.1, we are going to define a time rate of mass coming in from the left as $\dot{m}_{in}$ and the time rate of mass going out on the right as $\dot{m}_{out}$. Note that the units of these "mass flow rates" are, in SI units, kilograms per second. Thus, from the conservation of mass, we have:

$$\frac{dm_{sys}}{dt} = \dot{m}_{in} - \dot{m}_{out}, \tag{1.1}$$

where m_{sys} is the mass of fluid contained in the system (control volume). Equation 1.1 is nothing but our conservation principle that was discussed in the

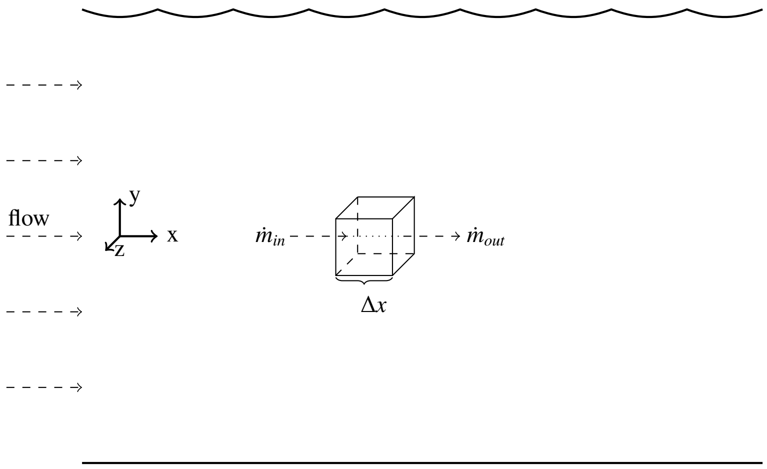

Figure 1.1 Mass passing through a box system fixed in space (control volume). The mass flow rate in ($\dot{m}_{in}$) minus the mass flow rate out ($\dot{m}_{out}$) is equal to the time rate of change of the mass in the system ($\frac{dm_{sys}}{dt}$).

previous section except now we are using variables instead of writing out the conservation principles in words.

Notice that the derivative on the left-hand side is used to denote the change of the mass of the fluid in the control volume with respect to time (t), whereas the dots ($\cdot$) above the mass flow rates denote the rate of mass entering or leaving the control volume in a given unit of time. The use of the dots is a common convention used to denote the rate of a quantity entering or leaving a system.

Now suppose we were not given mass flow rates but instead were given velocity and density going in and out of the control volume, both of which might be more easily measurable. Can we modify Equation 1.1 to be in terms of velocity and density? To start, we might want to try to find an expression for the mass flow rate in terms of velocity and density.

To determine a mass flow rate given velocity and density, consider first how you would go about determining the amount of mass passing through a surface (which we can call m_{pass}). We know that mass is a volume ($\mathcal{V}$) multiplied by the density of the fluid (ρ). The volume of fluid passing through a surface in a given amount of time (Δt) can be found by multiplying the area of the surface the fluid passes through (A) by the distance the fluid travels during that time period. The distance the fluid travels is the velocity of the fluid going into or out of the surface (which we will call u in this case[2]) multiplied by the time passed: $u\Delta t$.

[2] The velocity in the x-direction is, by convention in fluid mechanics, given by the variable u.

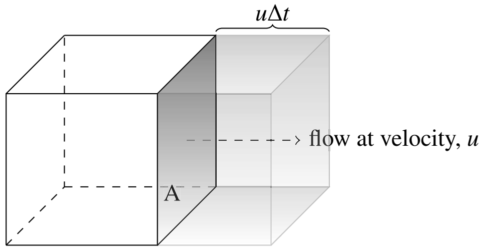

Figure 1.2 Illustration of how much volume of flow passes through a surface of area, A, in a given time, Δt. The volume is given by $u\Delta tA$, where $u\Delta t$ is the distance the flow has traveled. The particular surface in the schematic above (shaded in gray) is the surface for the outflow of our box system. A similar process can be done for an inflow surface.

A schematic of this process is given in Figure 1.2. Thus the amount of mass passing through the particular surface in Figure 1.2 is:

$$m_{pass} = \rho u \Delta t A.$$

To be a little more clear, we can make some comments on the above equation:

$$m_{pass} = \overbrace{\rho}^{\text{density}} \overbrace{\underbrace{u\Delta t}_{\substack{\text{distance fluid} \\ \text{travels in } \Delta t}} \; A}^{\substack{\text{volume of flow} \\ \text{that passes a} \\ \text{surface in time, } \Delta t}} .$$

Dividing by Δt gives a simple expression for the time rate of mass (mass flow rate) passing through a surface. Defining $\dot{m}$ as a mass flow rate then gives us the following expression:

$$\dot{m} = \frac{m_{pass}}{\Delta t} = \rho u A. \tag{1.2}$$

The product, ρu, is called the mass flux, or more specifically in this scenario, the mass flux in the x-direction.

It is important to recognize that this expression is not a general expression for the mass flow rate. For one thing, it assumes that the velocity of the flow is in a direction such that the flow comes straight out of the surface, A, and not at an angle. In addition, the velocity is inherently assumed to be uniform throughout the surface it is passing. We will come back to a more general form of mass flow rate. For now, we can just use Equation 1.2 in our quest to arrive at an equation for the mass balance in terms of density and velocity.

Plugging Equation 1.2 into Equation 1.1 yields:

$$\frac{dm_{sys}}{dt} = (\rho u A)_{in} - (\rho u A)_{out}. \tag{1.3}$$

Focusing now on the left side of Equation 1.3, we can write m_{sys} as the density of the fluid multiplied by the volume of our box system ($A\Delta x$), where Δx is the length of the box system in the x-direction. This would yield:

$$\frac{d}{dt}(\rho A \Delta x) = (\rho u A)_{in} - (\rho u A)_{out}.$$

Pulling out the $A\Delta x$ from the time derivative (since it is a constant) and dividing it out leads to:

$$\frac{\partial \rho}{\partial t} = \frac{(\rho u)_{in} - (\rho u)_{out}}{\Delta x}. \tag{1.4}$$

The reason for switching to a partial derivative on the left-hand side of Equation 1.4 will become apparent when we deal with the more general case later in the chapter. For now, just recognize that the density is dependent not only on time but also on space (i.e., is also a function of x), hence the partial derivative. We can rearrange Equation 1.4 to get all terms on the left of the equal sign:

$$\frac{\partial \rho}{\partial t} + \frac{(\rho u)_{out} - (\rho u)_{in}}{\Delta x} = 0.$$

Recognize that the second term on the left-hand side of the above equation is just the change in ρu over the change in x. This is the slope of ρu in the x-direction. Taking the limit of this slope as $\Delta x \to 0$ makes the second term become a derivative, that is:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0. \tag{1.5}$$

Equation 1.5 is a one-dimensional version of the continuity equation in Cartesian coordinates. It states, in effect, that the change of density (or mass per volume) with respect to time plus the spatial change of the mass flux (ρu) is equal to zero. It comes directly from a simple mass conservation equation, Equation 1.1. As you can see, even in the one-dimensional case, this equation is a complicated differential equation (i.e., an equation that includes derivatives). In particular, it is what is known as a partial differential equation (PDE) because the dependent variable (namely density, ρ) is a function of both time (t) and space (x). Thus, the derivatives seen in the differential equation are partial derivatives. As we will see, the governing equations are often written as partial differential equations. Unfortunately, PDEs are quite difficult to solve, and thus many times the solution of them requires the use of a computer. We will discuss this further at various times as we proceed.

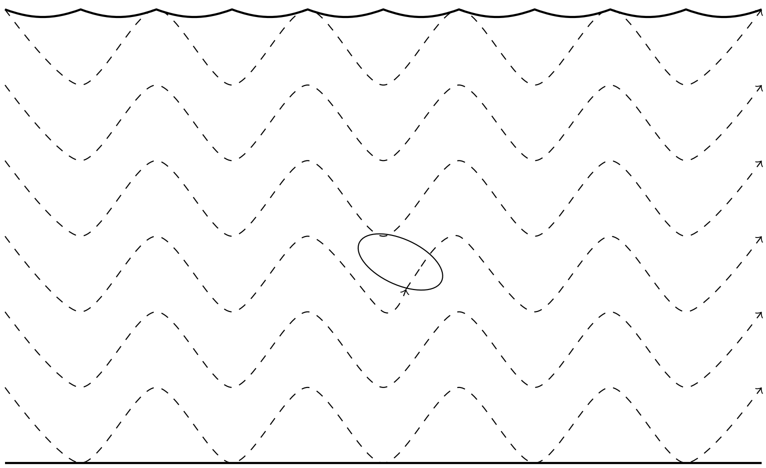

Figure 1.3 Flow passing through a control volume shaped as an ellipsoid.

We have looked at a simple one-dimensional situation. We can now extend the idea to obtain a general continuity equation for multiple dimensions.

1.3 The Continuity Equation

In this section, we are going to take a look at the continuity equation in a more general form. Instead of a 1-D flow situation that was studied earlier, consider the control volume in the middle of the wavy flow as shown in Figure 1.3. The shape of the control volume is arbitrary. Sometimes in fluid mechanics and continuum mechanics texts, such arbitrary volumes are called potatoes and are thus sometimes shaped as such. The shape we will use for our control volume will just be an ellipsoid. Suppose flow is passing through our control volume. We can calculate the change in mass contained in our control volume with respect to time (i.e., the left-hand side of Equation 1.1) by taking the time derivative of the mass of fluid contained in the fixed control volume. The mass of the fluid inside the control volume (m_{sys}), is given by the volume integral of density (ρ):

$$m_{sys} = \iiint_{\mathcal{V}} \rho d\mathcal{V}. \tag{1.6}$$

You might be wondering why mass is not just the density multiplied by the volume of our ellipsoid ($\mathcal{V}$). The reason for the volume integral is that we are now going to consider that the density may vary within the control volume.

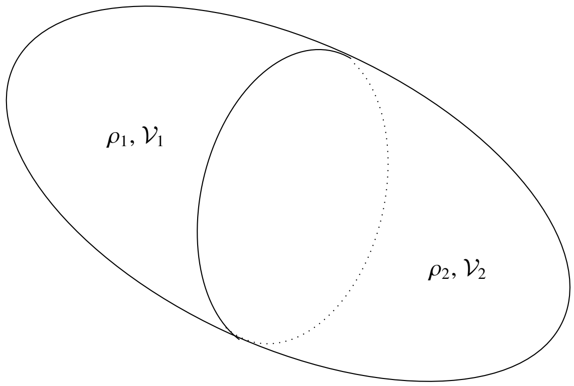

Figure 1.4 The control volume split into two pieces, each with its own volume and density. The total mass of the system in this case is given by: $m_{sys} = \rho_1\mathcal{V}_1 + \rho_2\mathcal{V}_2$. If we were to break up the system into infinitesimal volumes, each with a volume of $d\mathcal{V}$, the total mass would then be an infinite summation (an integral) of the $\rho d\mathcal{V}$s over the whole volume, that is, $\iiint_{\mathcal{V}} \rho d\mathcal{V}$.

Therefore, the integral takes into account a varying density within the volume. Thus, if the system is broken into two pieces, as shown in Figure 1.4, where one piece has a density of ρ_1 and a volume of $\mathcal{V}_1$ and the other piece has a density of ρ_2 and a volume of $\mathcal{V}_2$, then the mass of the system will just be:

$$m_{sys} = \rho_1\mathcal{V}_1 + \rho_2\mathcal{V}_2.$$

Since our control volume could theoretically be broken up into an infinite number of infinitesimal "pieces," all with a separate density and infinitesimal volume ($d\mathcal{V}$), an integral is used to "sum" over all of the little masses (i.e., $\rho d\mathcal{V}$) to get the total mass of the system, m_{sys}.

Next, we need to determine the rate of mass coming in or going out of the control volume. Unlike our 1-D version where the velocity of the flow was only in the x-direction, this time the velocity of the flow coming out of the surface might be at an angle and not necessarily coming "straight out" of the surface. We will denote the velocity of the flow by the vector $\vec{V}$. Generally, in Cartesian coordinates, the velocity vector can be broken up into components (with $\hat{i}$, $\hat{j}$, and $\hat{k}$ being the base vectors in the x-, y-, and z-directions, respectively) via:

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k} \tag{1.7}$$

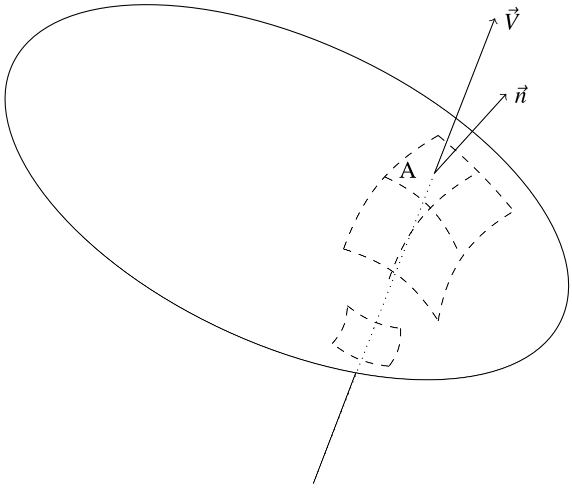

Figure 1.5 The ellipsoid control volume broken up into area segments (only a few area segments are shown). The velocity vector of the flow ($\vec{V}$) coming out of one of the area segments is shown, as is the unit normal ($\vec{n}$). The surface area (A) is also provided.

Or in column vector form:

$$\vec{V} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{1.8}$$

where u is the x-component of velocity, v is the y-component of velocity, and w is the z-component of velocity. This naming convention for the velocity components might seem odd but is very standard in fluid dynamics texts for Cartesian coordinates.

The surface of our control volume can be broken up into little pieces (area segments), each one having a unique velocity vector denoting the velocity of the flow going in or out. Figure 1.5 provides an illustration. There are a few area segments shown, with one in particular that is marked with a surface area of A. The area should be small enough such that we can define what is called an outward unit normal (or unit normal or outward normal) for that given surface, $\vec{n}$. The unit normal is a vector of unit length that is perpendicular, or normal, to the area segment and points away (hence outward) from the volume. In Cartesian coordinates, the unit normal can be written as:

$$\vec{n} = n_x\hat{\imath} + ny\hat{\jmath} + n_z\hat{k} \tag{1.9}$$