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Jan van Neerven holds an Antoni van Leeuwenhoek professorship at Delft University of Technology. Author of four books and more than 100 peer-reviewed articles, he is a leading expert in functional analysis and operator theory and their applications in stochastic analysis and the theory of partial differential equations.

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Functional Analysis

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Preface

This book is based on notes compiled during the many years I taught the course "Applied Functional Analysis" in the first year of the master's programme at Delft University of Technology, for students with prior exposure to the basics of Real Analysis and the theory of Lebesgue integration. Starting with the basic results of the subject covered in a typical Functional Analysis course, the text progresses towards a treatment of several advanced topics, including Fredholm theory, boundary value problems, form methods, semigroup theory, trace formulas, and some mathematical aspects of Quantum Mechanics. With a few exceptions in the later chapters, complete and detailed proofs are given throughout. This makes the text ideally suited for students wishing to enter the field.

Great care has been taken to present the various topics in a connected and integrated way, and to illustrate abstract results with concrete (and sometimes nontrivial) applications. For example, after introducing Banach spaces and discussing some of their abstract properties, a substantial chapter is devoted to the study of the classical Banach spaces $C(K)$, $L^p(\Omega)$, $M(\Omega)$, with some emphasis on compactness, density, and approximation techniques. The abstract material in the chapter on duality is complemented by a number of nontrivial applications, such as a characterisation of translation-invariant subspaces of $L^1(\mathbb{R}^d)$ and Prokhorov's theorem about weak convergence of probability measures. The chapter on bounded operators contains a discussion of the Fourier transform and the Hilbert transform, and includes proofs of the Riesz–Thorin and Marcinkiewicz interpolation theorems. After the introduction of the Laplace operator as a closable operator in L^p, its closure Δ is revisited in later chapters from different points of view: as the operator arising from a suitable sesquilinear form, as the operator $-\nabla^\star\nabla$ with its natural domain, and as the generator of the heat semigroup. In parallel, the theory of its Gaussian analogue, the Ornstein–Uhlenbeck operator, is developed and the connection with orthogonal polynomials and the quantum harmonic oscillator is established. The chapter on semigroup theory, besides developing the general theory, includes a detailed treatment of some important examples such as the heat semigroup, the Poisson semigroup, the Schrödinger group, and the wave group. By presenting the material in this integrated manner, it is hoped that the reader will appreciate Functional Analysis as a subject that, besides having its own depth and beauty, is deeply connected with other areas of Mathematics and Mathematical Physics.

In order to contain this already lengthy text within reasonable bounds, some choices had to be made. Relatively abstract subjects such as topological vector spaces, Banach algebras, and $C^\star$-algebras are not covered. Weak topologies are introduced *ad hoc*, the use of distributions in the treatment of weak derivatives is avoided, and the theory of Sobolev spaces is developed only to the extent needed for the treatment of boundary value problems, form methods, and semigroups. The chapter on states and observables in Quantum Mechanics is phrased in the language of Hilbert space operators.

A work like this makes no claim to originality and most of the results presented here belong to the core of the subject. Not just the statements, but often their proofs too, are part of the established canon. Most are taken from, or represent minor variations of, proofs in the many excellent Functional Analysis textbooks in print.

Special thanks go to my students, to whom I dedicate this work. Teaching them has always been a great source of inspiration. Arjan Cornelissen, Bart van Gisbergen, Sigur Gouwens, Tom van Groeningen, Sean Harris, Sasha Ivlev, Rik Ledoux, Yuchen Liao, Eva Maquelin, Garazi Muguruza, Christopher Reichling, Floris Roodenburg, Max Sauerbrey, Cynthia Slotboom, Joop Vermeulen, Matthijs Vernooij, Anouk Wisse, and Timo Wortelboer pointed out many misprints and more serious errors in earlier versions of this manuscript. The responsibility for any remaining ones is of course with me. A list with errata will be maintained on my personal webpage. I thank Emiel Lorist, Lukas Miaskiwskyi, and Ivan Yaroslavtsev for suggesting some interesting problems, Jock Annelle and Jay Kangel for typographical comments, and Francesca Arici, Martijn Caspers, Tom ter Elst, Markus Haase, Bas Janssens, Kristin Kirchner, Klaas Landsman, Ben de Pagter, Pierre Portal, Fedor Sukochev, Walter van Suijlekom, and Mark Veraar for helpful discussions and valuable suggestions.

A significant portion of this book was written in the extraordinary circumstances of the global pandemic. The sudden decrease in overhead and the opportunity of working from home created the time and serenity needed for this project. Paraphrasing the epilogue of W. F. Hermans's novel *Onder Professoren* (Among Professors), the book was written entirely in the hours otherwise spent on departmental meetings, committee meetings, evaluations, accreditations, visitations, midterms, reviews, previews, etcetera, and so forth. All that precious time has been spent in a very useful way by the author.

Delft, April 2022

In the present corrected version we have fixed numerous small misprints, a few misformulations and editing errors, as well as a small number of mathematical oversights. In some proofs, additional details have been written out, and some arguments have been streamlined. I thank Jan Maas for some valuable suggestions in this direction.

Delft, May 2023

Notation and Conventions

We write $\mathbb{N} = \{0,1,2,\ldots\}$ for the set of nonnegative integers, and $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ for the sets of integer, rational, real, and complex numbers. Whenever a statement is valid both over the real and complex scalar field we use the symbol $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$. Given a complex number $z = a + bi$ with $a, b \in \mathbb{R}$, we denote by $\overline{z} = a - bi$ its complex conjugate and by $\operatorname{Re} z = a$ and $\operatorname{Im} z = b$ its real and imaginary parts. We use the symbols $\mathbb{D}$ and $\mathbb{T}$ for the open unit disc and the unit circle in the complex plane, respectively. The indicator function of a set A is denoted by $\mathbf{1}_A$. In the context of metric and normed spaces, $B(x;r)$ denotes the open ball with radius r centred at x. The interior and closure of a set S are denoted by S° and $\overline{S}$, respectively. We write $S' \subseteq S$ to express that S' is a subset of S. The complement of a set S is denoted by $\complement S$ when the larger ambient set, of which S is a subset, is understood. We write $|x|$ both for the absolute value of a real number $x \in \mathbb{R}$, the modulus of a complex number $x \in \mathbb{C}$, and the euclidean norm of an element $x = (x_1, \ldots, x_d) \in \mathbb{K}^d$. When dealing with functions f defined on some domain f, we write $f \equiv c$ on $S \subseteq D$ if $f(x) = c$ for all $x \in S$. The null space and range of a linear operator A are denoted by $\mathsf{N}(A)$ and $\mathsf{R}(A)$ respectively. When A is unbounded, its domain is denoted by $\mathsf{D}(A)$. A comprehensive list of symbols is contained in the index.

Unless explicitly otherwise stated, the symbols X and Y denote Banach spaces and H and K Hilbert spaces. In order to avoid frequent repetitions in the statements of results, these spaces are always thought of as being given and fixed. Conventions with this regard are usually stated at the beginning of a chapter or, in some cases, at the beginning of a section. The same pertains to the choice of scalar field. In Chapters 1–5, the scalar field $\mathbb{K}$ can be either $\mathbb{R}$ or $\mathbb{C}$, with a small number of exceptions where this is explicitly stated, such as in our treatment of the Hahn–Banach theorem, the Fourier transform, and the Hilbert transform. From Chapter 6 onwards, spectral theory and Fourier transforms are used extensively and the default choice of scalar field is $\mathbb{C}$. In many cases, however, statements not explicitly involving complex numbers or constructions involving them admit counterparts over the real scalars which can be obtained by simple complexification arguments. We leave it to the interested reader to check this in particular instances.