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T. W. Körner , Foreword by Terence Tao
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PART 1
FOURIER SERIES

1
INTRODUCTION

We work on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ i.e. the real line mod 2π. Thus if $\theta \in \mathbb{T}$ then $\theta + 2\pi = \theta$. (Or see, if you must, Appendix A.)

Suppose f is a Riemann integrable function from $\mathbb{T}$ to $\mathbb{C}$ or to $\mathbb{R}$. Then we define the Fourier coefficients of f by

$$\hat{f}(r) = (2\pi)^{-1}\int_0^{2\pi} f(t)\exp(-irt)dt = (2\pi)^{-1}\int_{\mathbb{T}} f(t)\exp(-irt)dt.$$

Mathematicians of the eighteenth century (Bernoulli, Euler, Lagrange, etc.) knew 'experimentally' that for some simple functions

$$S_n(f,t) = \sum_{-n}^{n} \hat{f}(r)\exp irt \to f(t) \quad \text{as } n \to \infty.$$

Fourier claimed that this was always true and in a book of outstanding importance in the history of physics (*Théorie Analytique de la Chaleur*) showed how formulae of the kind $\sum_{-\infty}^{\infty} \hat{f}(r)\exp irt$ could be used to solve linear partial differential equations of the kind which dominated 19th century physics.

After several mathematicians (including Cauchy) had produced more or less fallacious proofs of convergence, Dirichlet took up the problem. In a paper which set up new and previously undreamed of standards of rigour and clarity in analysis, he was able to prove convergence under quite general conditions. For example, the following theorem is a consequence of his results.

Theorem 1.1. *If f is continuous and has a bounded continuous derivative except, possibly, at a finite number of points then $S_n(f,t) = \sum_{-n}^{n} f(r)\exp irt \to f(t)$ as $n \to \infty$ at all points t where f is continuous.*

However, it turned out that the conditions on f could not be relaxed indefinitely for Du Bois-Reymond constructed the following counter example.

Example 1.2. *There exists a continuous function f such that* $\limsup_{n\to\infty} S_n(f,0) = \infty$.

(Theorem 1.1 will be proved in Chapter 15 in the case f everywhere continuous and in Chapter 16 in the general case, whilst Example 1.2 will be constructed in Chapter 18.)

A new question was thus posed.

Question 1.3. *If f is a continuous function from $\mathbb{T}$ to $\mathbb{C}$ then, given the Fourier coefficients $\hat{f}(r)[r\in\mathbb{Z}]$ of f, can we find $f(t)$ for $t\in\mathbb{T}$?*

To the surprise of everybody Fejér (then aged only 19) showed that the answer is yes. He started from the observation that if a sequence $s_0, s_1, \ldots$ is not terribly well behaved, its behaviour may be improved by considering averages s_0, $(s_0+s_1)/2$, $(s_0+s_1+s_2)/3, \ldots$ This had, of course, been known since the time of Euler, but the first person to study the phenomenon in detail was Cesàro about ten years before Fejér's discovery.

Lemma 1.4. (i) *If $s_n\to s$ then $(n+1)^{-1}\sum_{j=0}^n s_j\to s$.*
(ii) *There exist sequences s_n such that s_n does not tend to a limit but $(n+1)^{-1}\sum_{j=0}^n s_j$ does.*

(Thus the 'Cesàro limit' exists, and is equal to the usual limit, whenever the usual limit exists and may exist even if the usual limit does not.)

Proof. (i) Let $\varepsilon>0$ be given. Since $s_n\to s$ we can find an $N(\varepsilon)$ such that $|s_n-s|\leqslant\varepsilon/2$ for $n\geqslant N(\varepsilon)$. Set $A=\sum_{j=1}^{N(\varepsilon)}|s_j-s|$ and choose $M(\varepsilon)\geqslant N(\varepsilon)$ such that $M(\varepsilon)\geqslant 2A\varepsilon^{-1}$. Then if $n\geqslant M(\varepsilon)$,

$$\left|(n+1)^{-1}\sum_{j=0}^n s_j - s\right| = (n+1)^{-1}\left|\sum_{j=0}^n (s_j-s)\right|$$
$$\leqslant (n+1)^{-1}\sum_{j=0}^n |s_j-s| = (n+1)^{-1}\left(\sum_{j=0}^{N(\varepsilon)}|s_j-s| + \sum_{j=N(\varepsilon)+1}^{n}|s_j-s|\right)$$
$$\leqslant (n+1)^{-1}(A+(n-N(\varepsilon))\varepsilon/2)\leqslant (n+1)^{-1}((n+1)\varepsilon/2+(n+1)\varepsilon/2)=\varepsilon.$$

(ii) Let $s_n=(-1)^n$ so that s_n fails to converge as $n\to\infty$. Then

$$\left|(n+1)^{-1}\sum_{j=0}^n s_j\right| = (n+1)^{-1}\left|\sum_{j=0}^n s_j\right|\leqslant (n+1)^{-1}\to 0$$

as $n\to\infty$, so $(n+1)^{-1}\sum_{j=0}^n s_j\to 0$ as $n\to\infty$. ■

Fejér saw that although partial sums $S_n(f,t)=\sum_{r=-n}^n\hat{f}(r)\exp irt$ could fail to converge their averages

$$\sigma_n(f,t)=\frac{1}{n+1}\sum_{j=0}^n S_j(f,t)=\sum_{r=-n}^n\frac{n+1-|r|}{n+1}\hat{f}(r)\exp irt$$

might behave rather better and that a Cesàro limit could take the place of the usual limit.

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Theorem 1.5. (i) *If $f:\mathbb{T}\to\mathbb{C}$ is Riemann integrable then, if f is continuous at t,*

$$\sigma_n(f,t)=\sum_{-n}^{n}\frac{n+1-|r|}{n+1}\hat{f}(r)\exp irt\to f(t).$$

(ii) *If $f:\mathbb{T}\to\mathbb{C}$ is continuous then*

$$\sigma_n(f,t)=\sum_{-n}^{n}\frac{n+1-|r|}{n+1}\hat{f}(r)\exp irt\to f(t)$$

uniformly on $\mathbb{T}$.

(Any reader discouraged by Fejér's precocity should note that a few years earlier his school considered him so weak in mathematics as to require extra tuition.)