Introduction

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The year 2022 is the 100th anniversary of Olga A. Ladyzhenskaya, a famous Russian mathematician who played an outstanding role in developing the modern theory of partial differential equations (PDEs) and their applications including the qualitative theory of PDEs, infinite-dimensional dynamical systems, mathematical problems of hydrodynamics and nonlinear problems of mathematical physics.

The book we are presenting is the expanded version of Ladyzhenskaya's lecture notes for the course of lectures given by her at the Accademia Nazionale dei Lincei in 1991 and the aim of this book is to give a brief introduction to the mathematical foundations of the theory of infinite-dimensional dynamical systems and their attractors, with applications to several classes of dissipative nonlinear PDEs.

Ladyzhenskaya came to attractors from her favourite topic, the mathematical theory of viscous incompressible fluids, by trying to understand the nature of turbulence by interpreting the Navier–Stokes system as an infinite-dimensional dynamical system and using/extending the ideas and methods of classical dynamics. Such an interpretation became possible due to her fundamental result on the uniqueness of solutions for the 2D Navier–Stokes problem, proven in 1958, see [62] (see also [63] for a more detailed exposition). Precisely, it has been proved there that the initial boundary value problem

$$\begin{cases} \partial_t v + \sum_{k=1}^{2} v_k v_{x_k} + \nabla p = \nu \Delta v + f, \quad \operatorname{div} v = 0, \quad t > 0, \ x \in \Omega, \\ v = 0, \quad x \in \partial\Omega, \quad v(x,0) = v_0(x), \ x \in \Omega, \end{cases} \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^2$ possesses a unique global solution $v(t)$ for all external forces f and initial conditions v_0 belonging to a properly chosen function space.

When this global well-posedness is established one can define a solution semigroup V_t of this problem in the proper phase space H and treat the Navier–Stokes equations as an infinite-dimensional *dynamical system* V_t acting on this phase space H (analogously with the classical qualitative theory of dynamical systems generated by ODEs). In her seminal paper [65], Ladyzhenskaya constructs a special set $\mathcal{M}$ in the phase space H (roughly speaking, the set of square integrable solenoidal vector fields) with the following properties:

- $\mathcal{M}$ is *invariant*: $V_t(\mathcal{M}) = \mathcal{M}$, i.e. $\mathcal{M}$ consists of trajectories of the dynamical system V_t;
- all trajectories are *attracted* by $\mathcal{M}$, i.e. any trajectory that started from a bounded set B comes to an ε-neighbourhood of $\mathcal{M}$ after a finite time $T(\varepsilon, B)$ and remains there;
- $\mathcal{M}$ is *compact* in H and is therefore in a sense finite-dimensional.

This set is constructed as an ω-limit set of the absorbing ball of the semigroup V_t and is *exactly* what is nowadays called *a global attractor*.

> Ladyzhenskaya wrote in [69] about the reasons that prompted her to study attractors: I tried to understand what the experimenter can observe after a very long (infinite) period of time. At the same time, I started from the statement widespread among physicists that the solutions of dissipative systems 'forget' their initial data and are 'formed' under the influence of constantly (stationary) acting factors. In the literal sense of the word, this, of course, is not true, because in a deterministic system (I had in mind only such systems, and first of all the two-dimensional Navier–Stokes equations, for which the global unique solvability of the initial-boundary value problems has been proved) solutions are determined by their initial data (as well as boundary conditions and external forces, which are considered fixed and independent of time). But in the course of time, the solution may move far away from them and, in this sense, forget them. And I asked myself the question: what is the part of the phase space to which solutions are attracted and what is the dynamics on this part?

These words are very similar to the modern description of so-called *deterministic chaos*, an extremely interesting and important, and in a sense still 'mysterious' phenomenon which allows a deterministic system to demonstrate random behaviour. Note that this phenomenon has been observed in hydrodynamics and weather prediction by Lorentz [78] and is nowadays considered to be one of the characteristic features of turbulence.

The prominent ideas of Ladyzhenskaya inspired many brilliant mathematicians to switch to this area and led to the development of a general theory of infinite-dimensional dynamical systems governed by dissipative PDEs (also known as *attractor theory*), which is the main subject of the book we are presenting.

The book consists of two parts. Part I, comprising Chapters 1–4, is focused on the theory of global attractors for semigroups defined on a complete metric space X. Part II containing Chapters 5–7 is about particular semigroups generated by initial boundary value problems for 2D Navier–Stokes equations, nonlinear parabolic equations, damped nonlinear wave equations, etc.

In Chapter 1, a number of basic notions are defined. Since some of them differ from the modern terminology, we reproduce them and indicate differences.

- A semigroup $V_t : X \to X$ is called a *bounded semigroup* if for each bounded set $B \subset X$, the orbit set $\gamma^+(B) := \cup_{x \in B}\{V_t x,\ t \in \mathbb{R}^+\}$ is bounded.
- A set $A \subset X$ *attracts* a set $M \in X$ if for each $\varepsilon > 0$ there exists $T = T(\varepsilon, M)$ such that $V_t(M) \subset O_\varepsilon(A)$ for all $t \geq T$, where $O_\varepsilon(A)$ is an ε-neighbourhood of the set A.
- A set A attracting each point of X is called a *global attractor* of the semigroup V_t and a set that attracts each bounded set $B \subset X$ is called the *global B-attractor* of the semigroup V_t.
- A semigroup is called *pointwise dissipative* if it has a bounded global attractor and it is called *B-dissipative* if it has a bounded global B-attractor.
- A ball B_R is called an *absorbing* set of the semigroup V_t if for each bounded set $B \subset X$ there exists $t_0(B)$ such that $V_t(B) \subset B_R$, $\forall t \geq t_0(B)$.

In modern terminology, a semigroup or dynamical system is called *dissipative* if it possesses a bounded absorbing ball. Moreover, a global B-attractor is nowadays simply known as a global attractor, and Ladyzhenskaya's 'global attractor' is known as a pointwise attractor. Thus, on the one hand, the attractor $\mathcal{M}$ captures all of the non-trivial limit dynamics of the system in question as time goes to infinity, while, on the other hand, it is essentially smaller than the initial phase space. In particular, in many cases, this attractor has a finite dimension, so there is a tremendous reduction of the effective 'degrees of freedom' (from infinite to finite), which in turn allows us to use the ideas and methods of classical dynamics to investigate the dynamics of PDEs.

The main objects of study in Chapter 2 are so-called *semigroups of class* $\mathcal{K}$, i.e. semigroups that for each $t > 0$ map any bounded set $B \subset X$ to a precompact set $V_t(B)$. Two important statements, Theorem 2.2 and Theorem 2.3, are proved in the chapter. Conceptually, they show the following:

- if a continuous semigroup $V_t : X \to X$ is of class $\mathcal{K}$ and B-dissipative or bounded and point-wise dissipative, then it possesses a minimal global B-attractor $\mathcal{M}$ which is compact, invariant and connected, provided X is connected;
- if $V_t : X \to X$ is a continuous semigroup of class $\mathcal{K}$, the orbit $\gamma^+(x)$ is bounded for each $x \in X$ and a 'good' Lyapunov function exists, then there is a minimal global attractor $\widehat{\mathcal{M}}$ that consists of the stationary points Z of the semigroup. In other words, as time tends to infinity any bounded trajectory of the semigroup converges to the set Z. Moreover, if Z is bounded, then the semigroup has a minimal global B-attractor $\mathcal{M}$. In addition, if Z is totally disconnected (e.g. if it consists of finitely many points), then the attractor $\mathcal{M}$ consists of equilibria Z and all heteroclinic orbits connecting different equilibria from Z.

Of course, there are many publications (see e.g. [6, 42, 44, 45, 85, 91] and references therein) where various theorems about attractors of semigroups of class $\mathcal{K}$ and/or their applications have been proven. To our knowledge, the first result on attractors of compact semigroups appeared in [11] and the results for the case where the Lyapunov function exists can be found in [42].

It is noteworthy that the class of semigroups possessing a global Lyapunov function is extremely important for the theory of attractors since it is the only known relatively large class for which we can say something reasonable about the structure of the attractor. The further development of this theory led to the concept of a *regular* attractor: a global attractor that consists of a finite union of finite-dimensional unstable manifolds of equilibria, see [8] and also [92] for the modern state of the art. Such attractors have a lot of good properties which usually fail for general attractors. For instance, they are robust with respect to perturbations and the rate of attraction to them of bounded sets is exponential. Moreover, under the 'generic' assumption that stable and unstable manifolds of equilibria intersect transversally (the Morse–Smale property), the dynamics on these attractors is also robust with respect to perturbations, see [17, 18, 33]. We also mention that the assumption that the equilibria set Z is totally disconnected can be partially removed using the Simon–Lojasiewicz technique, which gives the stabilization to a *single* equilibrium even in the case that Z is a continuous set, see e.g. [53].

The results of Chapter 2 are extended in the subsequent Chapter 3 to the class of *asymptotically compact* semigroups (semigroups of class $\mathcal{AK}$). These semigroups possess the property that each bounded sequence $V_{t_k}(x_k)$, with $\{x_k\} \subset X$ and $t_k \to \infty$, is a precompact set of X. As shown in Chapter 3, the main results proved for semigroups of class $\mathcal{K}$ remain valid for $\mathcal{AK}$-semigroups.

Asymptotically compact semigroups arise naturally in the study of non-parabolic equations (e.g. damped wave equations) that do not have the simultaneous smoothing property. Theorem 3.3 gives the main technical tool to study such equations. It claims that a semigroup V_t is asymptotically compact if it can be presented as a sum

$$V_t = U_t + W_t, \tag{2}$$

where the operators U_t are compact for every fixed t and W_t tend to zero uniformly with respect to bounded sets as time tends to infinity.

We mention that the theory of $\mathcal{AK}$ semigroups has undergone intense development during the last two decades and nowadays we have a number of effective methods for verifying their asymptotic compactness which do not require the splitting of the semigroup: for instance, the so-called *energy* method (see [10, 84]) or the methods based on compensated compactness or precompact semi-norms (see [24]).

Chapter 4 is devoted to upper bounds for the Hausdorff and fractal dimension of the attractors. The main result here is the proof of the classical volume contraction theorem, which states that a C^1-map V on a Hilbert space H contracts N-dimension volume in some neighbourhood of a compact invariant set $\mathcal{A}$, then the Hausdorff dimension of $\mathcal{A}$ does not exceed N, see Theorem 4.5. Combined with the Liouville formula for the evolution of k-dimensional volumes (see formula (4.28)) this yields one of the most popular modern methods for estimating the dimension of the attractors; it is especially effective for hydrodynamical problems. The analogous result for the fractal (box-counting) dimension is also given (see Theorem 4.6), but the estimate is essentially weaker and is not so elegant.

Note that while this key theorem for the Hausdorff dimension was obtained for the finite-dimensional case in [52] and for the infinite-dimensional case in [27], the same result for the fractal dimension was open for a long time and has only recently been established. A breakthrough on this problem came in the paper [49] where the result was obtained in the finite-dimensional case. Then it was extended to the infinite-dimensional diffeomorphisms in [13] and the final result in exactly the same formulation as for the Hausdorff dimension was obtained in [22]. Thus, nowadays there is no difference in estimating Hausdorff and fractal dimension via the volume contraction method.

The second part of the book is devoted to applications of the general theory developed in Part I to the classical equations of mathematical physics, namely, the 2D Navier–Stokes equations (Chapter 6) and damped wave equations (Chapter 7).

Chapter 6 contains two main results (Theorems 6.1 and 6.2) which give the upper bounds for the number of determining modes N and the fractal dimension $\dim_f(\mathcal{M})$ of the attractor of the 2D Navier–Stokes equation (1) in terms of the parameter ν. The estimates obtained for N are:

$$N \leq c_1\nu^{-4} + c_1' \quad \text{and} \quad N \leq c_2\nu^{-2}|\ln\frac{1}{\nu}| + c_2'$$

for the cases of no-slip (Dirichlet) and periodic boundary conditions, respectively. Upper bounds for the fractal dimension are obtained for the case of no-slip boundary conditions only and have the same form as for the corresponding determining modes:

$$\dim_H(\mathcal{M}) \leq c_3\nu^{-4} + c_3',$$

where c_i and c_i' above are some constants that are independent of ν.

Here we would like to make some important remarks. There is a heuristic conjecture (partially inspired by the conventional theory of turbulence of A.N. Kolmogorov from the one side and I. Prigogine's theory of dissipative structures from the other) that despite the infinite-dimensionality of the initial phase space, the limit dynamics of a dissipative system are finite-dimensional and can be effectively described by the evolution of finitely many parameters (the so-called order parameters in the terminology of I. Prigogine). One of the ultimate goals of the theory of attractors is to find a rigorous interpretation and justification of this conjecture. Historically, the first attempt at tackling this problem was made in the pioneering works of Foias and Prodi [34] and of Ladyzhenskaya [65] using precisely the method of determining modes.

To be more precise, it was proved that the limit dynamics of 2D Navier–Stokes equations are determined in a unique way if the evolution in time of the first N Fourier modes is known and if N is large enough. So in some sense these limit dynamics are determined by N parameters.

The further development of this theory went in the direction of generalizing the form of the determining modes and computing upper bounds that are as sharp as possible for the number N. In particular, analogous results have been obtained in [36] where Fourier modes are replaced by *nodes* (i.e. the values of the dependent variable at the nodes of some spatial grid). Later on, the notions of determining volume elements and so on were introduced and various upper bounds for the number N of determining elements in such systems were obtained for various dissipative PDEs (see e.g. [36, 38, 39] and references therein). The more general notion of determining functionals (or determining interpolant operators) as well as a unified approach for investigating parameters uniquely determining asymptotic behaviour of solutions to dissipative PDEs was introduced in [26] and developed in [23] (see e.g. [24] and references therein).

The key drawback of the described approach to the problem of finite-dimensionality is that the values of the 'slaved' higher modes at some fixed time t cannot be found in terms of the values of determining functionals at the same moment in time (one needs to know the values of determining functionals *for all times* in order to do this). In other words, the time evolution of the values of determining functionals are not governed by a system of ODEs and still can be infinite-dimensional. A closely related example here is a system of ODEs with delay where the 'number of parameters' is finite, but the dynamics can still be infinite-dimensional. This drawback inspired researchers to seek a stronger version of finite-dimensional reduction based on various dimensions of the attractor $\mathcal{M}$.

To the best of our knowledge, the first result on the finiteness of the Hausdorff dimension of a negatively invariant set in a Hilbert space was obtained by J. Mallet-Paret in [79] with applications to delay differential equations as well as to the 1D reaction diffusion equation. Based on this result, the first very rough estimate for the Hausdorff dimension of a global attractor of the 2D Navier–Stokes equations was obtained in [35].

Later on, in [67], Ladyzhenskaya proved the following result, which can also be treated as a generalization of the method of [79].

Theorem A1 *Let M be a bounded subset of a Hilbert space H, and let $V : M \to H$ be an operator such that $M \subset V(M)$ and satisfying the conditions*

$$\|V(v) - V(w)\| \le \ell \|v - w\|, \ \forall v, w \in H,$$

and

$$\|Q_N(V(v) - V(w))\| \le \delta \|v - w\|, \ \ \forall v, w \in H,$$

where $\ell > 0$, $\delta \in (0,1)$ are given numbers and Q_N is the orthogonal projection onto the subspace of co-dimension N. Then

$$\dim_H(M) \le N \ln\left(\frac{8\kappa^2\ell^2}{1-\delta^2}\right)\left[\ln\frac{2}{1+\delta^2}\right]^{-1},$$

where $\kappa > 0$ is an absolute constant.

Using this theorem, she got an estimate of the Hausdorff dimension of the global attractor for the 2D Navier–Stokes equations that grows exponentially in ν^{-1} in the case of no-slip boundary conditions (see also [51]).

The polynomial in ν^{-1} estimate (6.14) presented in this monograph is due to Babin and Vishik [7] (for the case of Hausdorff dimension) and Constantin and Foias [27] (for fractal dimension). However, this estimate is still far from being optimal and further progress in this direction is due to the use of so-called Lieb–Thirring inequalities, see [75]. Up to the moment, the best known upper bounds for the fractal dimension of the 2D Navier–Stokes equations can be found in [91]:

$$\dim_f(\mathcal{M}) \le CG, \quad G := \|f\|_{L^2}|\Omega|\nu^{-2},$$

where the modern value of the constant C is related to the constant C_{LT} in the corresponding Lieb–Thirring inequality via $C = \frac{C_{LT}^{1/2}}{2\sqrt{2}\pi}$. The explicit value of this constant is not available, but the best known analytic bound is $C_{LT} \le \frac{1}{2\sqrt{3}}$, see [22, 40].

For the case of periodic boundary conditions, the obtained upper bounds can be essentially improved to:

$$\dim_f(\mathcal{M}) \le cG^{2/3}(1 + \log G)^{1/3},$$

see [91]. Moreover, this estimate is in a sense sharp up to the logarithmic term: the lower bounds of the form

$$\dim_f(\mathcal{M}) \ge c'G^{2/3}$$

are attained on the properly constructed Kolmogorov flows, see [77]. Note also that no non-trivial lower bounds are known for the case of no-slip boundary conditions.

It is also noteworthy that Theorem A1 has an essential advantage in comparison with other methods, namely, that the differentiability of the corresponding semigroup is not required. For this reason, it can be applied to many classes of degenerate or singular problems as well as problems with supercritical nonlinearities where this differentiability is problematic or is difficult to prove, see [24, 25, 82] and references therein for further generalizations and applications. In particular, this theorem is very useful for estimating the dimensions of attractors for various problems related to non-Newtonian fluids in dimensions two or three, see e.g. [66, 71–73] and [80].

We also mention that the proper generalizations of the Ladyzhenskaya squeezing property used in Theorem A1 have been exploited later in order to demonstrate the existence of the so-called exponential attractor for various dissipative nonlinear PDEs, see [82] and references therein. The notion of an exponential attractor, which is somehow an intermediate object between the global attractor and an inertial manifold, was introduced in [30] in order to overcome major drawbacks of global attractors (sensitivity to perturbations and slow rate of attraction). Remarkably, the initial assumptions of [30] for the existence of such an object are very close to the assumptions of Theorem A1.

As we have already mentioned, Chapter 7 of the book is devoted to attractors for abstract semilinear damped wave equations. After the short introduction, the exposition begins (in Section 7.2) with the detailed analysis of the linear problem

$$\partial_t^2 v + \nu \partial_t v + Av = h, \;\; v(0) = v_0, \;\; v_t(0) = v_1, \tag{3}$$

in an abstract Hilbert space H. Here $A\colon D(A) \to H$ is a given positive self-adjoint linear operator with compact inverse and $h = h(t)$ is a given external force which may depend explicitly on time. The main result of the section is the existence and uniqueness theorem for the solutions of problem (3) in the appropriate energy spaces, which nowadays has become a standard technical tool for the study of more general *nonlinear* wave equations.

We would like to emphasize here that Ladyzhenskaya was one of the first mathematicians who applied functional analytic methods to study the solvability of initial boundary value problems for hyperbolic equations, which nowadays is classical (including the results presented in Section 7.2). Her first book [61] deals exactly with equation (3) in the particular case that A is a second order symmetric uniformly elliptic differential operator in a bounded domain $\Omega \subset \mathbb{R}^n$.

Section 7.3 is devoted to the study of the analogous problems for the nonlinear wave equation of the form

$$\partial_t^2 v + \nu \partial_t v + Av + f(v) = 0, \;\; v(0) = v_0, \;\; v_t(0) = v_1, \tag{4}$$

where the nonlinear function f is a smooth enough gradient (i.e. $f(v) = \mathcal{F}'(v)$ for some given non-negative potential $\mathcal{F}$) and is in a sense subordinate to the leading linear part Av.

In the subsequent Section 7.4 Ladyzhenskaya studied the differentiability of solutions of problem (4) with respect to initial data for which the application of the volume contraction method and estimation of the fractal dimension of the corresponding attractor is necessary.

This attractor (a global B-attractor in the terminology of the book) is constructed in Section 7.5 by verifying the associated solution semigroup V_t in the standard energy space belongs to the class $\mathcal{AK}$. This fact, in turn, is obtained with the help of a decomposition as in (2) where U_t is the solution operator that corresponds to the *linear* problem (4) with $f \equiv 0$. Finally, the result about the finite-dimensionality of the corresponding attractor is proved in the concluding Section 7.6.

We recall that the key model for the abstract damped wave equation (4) is the following dissipative PDE:

$$\begin{cases} \partial_t^2 v + \nu\partial_t v - \Delta v + f(v) = h, \ \ v\big|_{\partial\Omega} = 0, \\ v(x,0) = v_0(x), \quad \partial_t v(x,0) = v_1(x), \end{cases} \tag{5}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and $f \in C^1(\mathbb{R})$ satisfies the dissipativity conditions

$$f(s)s - \mathcal{F}(s) \ge -C, \ \ \mathcal{F}(s) := \int_0^s f(\tau)d\tau \ge -C, \ C > 0,$$

as well as the growth restriction

$$|f'(s)| \le M(1 + |s|^{p-1}),$$

where $p \ge 1$ is a given growth exponent and $M > 0$.

In particular, the results presented in the book hold for this equation when

$$p < p_{en-crit} := \frac{2}{n-2} \ \ (p \text{ may be arbitrarily large if } n = 1 \text{ or } n = 2).$$

This is the so-called sub-critical energy case where the nonlinearity f is strongly subordinated to the Laplacian in the standard energy space. Actually, the results of Chapter 7 concerning global well-posedness hold for the energy critical case $p = p_{en-crit}$, but the method of verifying the asymptotic compactness requires $p < p_{en-crit}$. We emphasize that this damped wave equation is not the only equation to which the results of Chapter 7 can be applied. Among other interesting examples are various versions of nonlinear plate equations, von Karman equations, etc., see [24, 25] for more details.

It is noteworthy that the class of wave equations of the form (5) is one of the most important classes of PDEs and has been studied intensively by many prominent mathematicians. The first results on global existence, uniqueness, and regularity of weak solutions to the Cauchy problem for some cases of equation (5) with $p \in [1,4)$ (in 3D case) are demonstrated in [55] and for the initial boundary value problem with $p \in [1,3]$ in [76] and [88]. Later on, the above results were proved for the Cauchy problem with quintic nonlinearity ([58], [50]) and for the initial boundary value problem with $p \in [3,5]$ in [15]. The supercritical case $p > 5$ is much more delicate and the uniqueness of energy solutions in this case is still an open problem. The recent result of Tao

[90] shows that finite time blow-up of smooth solutions may appear in *systems* of equations of type (5) in the supercritical case.

With regards to the attractors for equation (5), a number of papers appeared on this topic in the beginning of the 1980s, although some preliminary results in this direction had been obtained a little earlier by Ball [9] and Webb [93].

The existence of a global attractor for equation (5) in the standard energy phase space in the energy subcritical case $p < p_{en-crit}$ was obtained by Haraux [46] and Hale [43], see also [70], and estimates for the dimension of this attractor were obtained by Ghidaglia and Temam in [41] and Ladyzhenskaya [68].

The energy critical case $p = p_{en-crit}$ has been treated by Babin and Vishik in [8] (see also Arieta, Carvalho and Hale in [2]); the existence of a global attractor for the critical case was also obtained by Ladyzhenskaya [70], but in higher energy space only. The key idea of their method is to use the so-called dissipation integral together with a slightly delicate decomposition as in (2) in which both operators U_t and W_t are nonlinear. This decomposition also allowed them to establish the extra smoothness of the constructed global attractor. Note that the usage of the dissipation integral was actually a serious restriction which did not allow them to extend the method to non-autonomous external forces or unbounded domains. This restriction was later removed in [94].

The existence of a global attractor for problem (5) on a compact n-dimensional Riemann manifold without boundary (e.g. on a torus that corresponds to periodic boundary conditions) was established in [59] (see also [32] for the analogous result for $\Omega = \mathbb{R}^3$) under the assumption

$$p \in (1, p_{\text{crit}}), \quad p_{\text{crit}} := \frac{4}{n-2}.$$

The key new idea that allows them to shift the limit exponent is related to the so-called Strichartz estimates, which give control of the $L^4(0,T; L^{12}(\Omega))$-norm of the solution and this in turn leads to the uniqueness of the properly defined weak solution. Remarkably, as is pointed out in [59], the idea of using Strichartz estimates in the theory of attractors actually came from Ladyzhenskaya.

In contrast to this, the analogous result *in a bounded domain* remained open for a long time, because of the absence of Strichartz estimates for manifolds with non-empty boundary. The breakthrough in the theory of such estimates was made in the mid-2000s, see [12, 15] and references therein. By combining these results with the classical Pohozhaev–Morawetz identity, the global well-posedness of (5) with the critical growth exponent $p = 5$