Homogeneous Ordered Graphs, Metrically Homogeneous Graphs, and Beyond

Volume I: Ordered Graphs and Distanced Graphs

This is the first of two volumes by Professor Cherlin presenting the state of the art in the classification of homogeneous structures in binary languages and related problems in the intersection of model theory and combinatorics. Researchers and graduate students in the area will find in these volumes many far-reaching results and interesting new research directions to pursue.

In this volume, Cherlin develops a complete classification of homogeneous ordered graphs and provides a full proof. He then constructs a new family of metrically homogeneous graphs, a generalization of the usual homogeneity condition. A full classification conjecture for such graphs is presented, together with a general structure theory and applications to the classification conjecture. The text also includes introductory chapters giving an overview of the results and methods of both volumes, and an appendix surveying recent developments in the area. An extensive accompanying bibliography of related literature, organized by topic, is available online.

Gregory Cherlin is Distinguished Professor Emeritus at Rutgers University. He has worked on applications of model theory to algebra and combinatorics for half a century, and has published four books and over 100 articles on model theory and its applications.

LECTURE NOTES IN LOGIC

A Publication of The Association for Symbolic Logic

This series serves researchers, teachers, and students in the field of symbolic logic, broadly interpreted. The aim of the series is to bring publications to the logic community with the least possible delay and to provide rapid dissemination of the latest research. Scientific quality is the overriding criterion by which submissions are evaluated.

More information, including a list of the books in the series, can be found at http://aslonline.org/books/lecture-notes-in-logic/

LECTURE NOTES IN LOGIC 53

Homogeneous Ordered Graphs, Metrically Homogeneous Graphs, and Beyond

Volume I: Ordered Graphs and Distanced Graphs

GREGORY CHERLIN
Rutgers University, New Jersey

ASSOCIATION FOR SYMBOLIC LOGIC

CAMBRIDGE UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781009229692
DOI: 10.1017/9781009229661

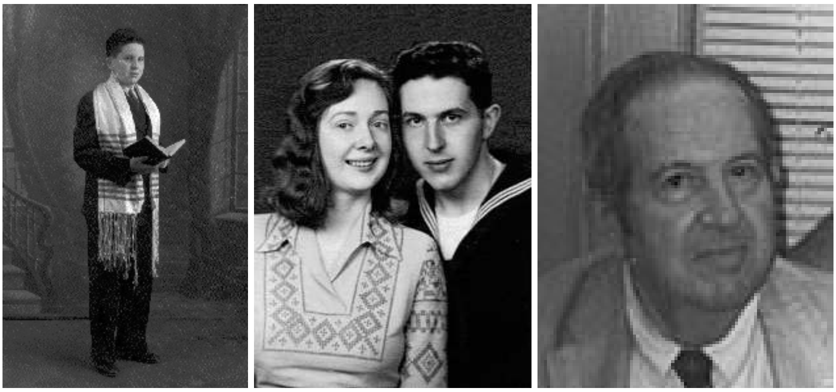
Association for Symbolic Logic
Richard A. Shore, Publisher
Department of Mathematics, Cornell University, Ithaca, NY 14853
http://aslonline.org

First published 2022

A catalogue record for this publication is available from the British Library.

ISBN – 2 Volume Set 978-1-009-23018-6 Hardback
ISBN – Volume I 978-1-009-22969-2 Hardback
ISBN – Volume II 978-1-009-22948-7 Hardback

In memory of George Cherlin, 1924–1992.
Enthralled by the beauty of mathematics,
ever mindful of its power for good or evil.



CONTENTS

Volume I

List of Tables xv

Abstract xvii

Preface xix

Acknowledgments xxix

Chapter 1. Results 1
1.1. Introduction 1
1.2. Fraïssé limits and amalgamation classes 2
1.3. The classification of countable homogeneous ordered graphs 5
1.4. The metrically homogeneous catalog 10
1.4.1. Homogeneous graphs and the graph Γ_1 12
1.4.2. Finite metrically homogeneous graphs 13
1.4.3. Tree-like graphs $T_{m,n}$ 14
1.4.4. 3-constrained amalgamation classes 15
1.4.5. Henson constraints and antipodal variants 15
1.4.6. Generic type 18
1.5. Part II: 3-constraint (we take up trigonometry) 19
1.6. Bipartite & infinite diameter cases 25
1.7. Toward a proof of the classification 27

Chapter 2. Methods 31
2.1. Homogeneous ordered vs. ordered homogeneous 31
2.1.1. Recognizing the underlying homogeneous graphs 33
2.1.2. The inductive framework 34
2.2. 3-constrained metrically homogeneous graphs 42
2.3. Reduction theorems 46
2.3.1. The bipartite case 46
2.3.2. The case of infinite diameter 47

2.4. Local analysis 48

Part I. Homogeneous ordered graphs

CHAPTER 3. THE CATALOG OF HOMOGENEOUS ORDERED GRAPHS 53
3.1. Preliminaries 53
3.2. The catalog 54
3.3. Homogeneous ordered graphs omitting $\vec{C}_3^+$ or $\vec{C}_3^-$ 59

CHAPTER 4. THE GENERICALLY ORDERED LOCAL ORDER 61
4.1. Statement of the problem 61
4.2. Permutations 62
4.3. $I_1 \to C_3$ and $C_3 \to I_1$ 70
4.4. Characterization of the generically ordered local order 77

CHAPTER 5. ORDERED HOMOGENEOUS GRAPHS: PLAN OF THE PROOF, PROPOSITIONS I–IX 83
5.1. Introduction 83
5.2. A simultaneous induction 85
5.3. From Propositions I–V to Propositions VI–IX 88
5.4. Ramsey theoretic arguments 90

CHAPTER 6. ORDERED HOMOGENEOUS GRAPHS: PROPOSITION I 95
6.1. Proof of Proposition IA 96
6.2. Some small configurations 100
6.3. Four vertices and two edges 103
6.4. An ample ordered 2-graph 107
6.5. 1-types over $\vec{I}_k$; Proposition IB 108

CHAPTER 7. ORDERED HOMOGENEOUS GRAPHS: PROPOSITION II 113
7.1. Closure under $\vec{K}_2 \perp X$ 113
7.2. $\mathcal{A}(n)$ and $[\vec{I}_1, \mathcal{A}(n-1)]$ 117
7.3. Proof of Proposition II 118

CHAPTER 8. ORDERED HOMOGENEOUS GRAPHS: PROPOSITION III 121
8.1. Ramsey 2-types 121
8.2. An inductive principle 122
8.3. Ramsey 2-types over $\vec{K}_2$ 126
8.4. Proof of Proposition III 136

CHAPTER 9. ORDERED HOMOGENEOUS GRAPHS: PROPOSITION IV 139
9.1. Introduction 139
9.2. Proof of Proposition IV_2 139
9.3. Proof of Proposition IV_n–A: closure under $\perp$ 144
9.4. Proof of Proposition IV_n–B 151

CHAPTER 10. ORDERED HOMOGENEOUS GRAPHS: PROPOSITION V 159
10.1. An auxiliary type 159
10.2. A reduction 163
10.3. Final preparations 167
10.4. Proof of Proposition V_n 171

Part II. Metrically homogeneous graphs

CHAPTER 11. METRICALLY HOMOGENEOUS GRAPHS: PRELIMINARIES 179
11.1. The main conjecture 179
11.2. The graphs $\Gamma^{\delta}_{K_1,K_2,C_0,C_1,\mathcal{S}}$ 181
11.3. 4-triviality 183
11.4. Smith's Theorem 185
11.5. Metrically homogeneous graphs: status 186

CHAPTER 12. ADMISSIBILITY ALLOWS AMALGAMATION 189
12.1. Introduction 189
12.2. 2-point amalgamation 190
12.3. Amalgamation strategy & bipartite case 191
12.4. Statements of the main lemmas 193
12.5. Proofs of the main lemmas 196
12.6. Main Theorem I: amalgamation 206

CHAPTER 13. TRIANGLE CONSTRAINTS AND 4-TRIVIALITY 213
13.1. The Main Theorem, Part II: 4-triviality 213
13.2. Recovering the parameters 215
13.3. The interpolation property 216
13.4. Small even perimeter 219
13.5. An inductive lemma 219
13.6. Triangles of even perimeter 224
13.7. Triangles of odd perimeter 225
13.8. Identification of $\mathcal{A}$ 231
13.9. Parameters for non-generic type 232

CHAPTER 14. AMALGAMATION REQUIRES ADMISSIBILITY 235
14.1. K_1 and K_2 236
14.2. The low case: $C \leq 2\delta + K_1$ 237
14.3. The high case: $C > 2\delta + K_1$ 249

CHAPTER 15. LOCAL ANALYSIS 263
15.1. Introduction 263
15.2. The type of Γ_i 265
15.3. Local connectivity 272
15.4. Local primitivity 276

Cambridge University Press & Assessment
978-1-009-22969-2 — Homogeneous Ordered Graphs, Metrically Homogeneous Graphs, and Beyond
Gregory Cherlin
Frontmatter
More Information

15.5. The case of Γ_δ bipartite 280
15.6. Theorem 1.7.1 and Proposition 1.7.2 282
15.7. An application 284
15.8. The interpolation property revisited 285

CHAPTER 16. THE BIPARTITE CASE 289
16.1. Preliminary analysis 290
16.2. Realization of triangles 292
16.3. Embedding lemma: δ even or infinite 296
16.4. Henson constraints: δ finite and odd 301
16.5. A reduction lemma for δ finite and odd 304
16.6. The embedding lemma for δ odd, finite 306
16.7. Conclusion of the proof 311

CHAPTER 17. INFINITE DIAMETER 313
17.1. Introduction 313
17.2. Toward Theorem 1.6.1 314
17.3. The cases $K_1 \leq 2$ 317
17.4. General K_1 317

APPENDIX A. SOME RECENT ADVANCES 321
A.1. Homogeneous multi-orders 322
A.2. Metrically homogeneous graphs 325
A.2.1. Toward the classification conjecture 325
A.2.2. Canonical completion of partial substructures 329
A.2.3. The twisted automorphism group 340

REFERENCES FOR VOLUME I 345

INDEX 351

Volume II

CONTENTS . . . vii

LIST OF TABLES . . . xv

ABSTRACT . . . xvii

PREFACE TO VOLUME II . . . xix

ACKNOWLEDGMENTS . . . xxiii

CHAPTER 18. CLASSIFICATION PROBLEMS FOR SMALL BINARY LANGUAGES . . . 1
18.1. Lachlan's classification problem . . . 3
18.2. Standard binary homogeneous structures . . . 5
18.3. Concrete classification problems: overview . . . 7
18.3.1. Classification problems for small languages . . . 7
18.3.2. The main 3-constrained homogeneous 3-m.g.'s . . . 8
18.3.3. The main 3-constrained homogeneous 2-m.t.'s . . . 9
18.3.4. The story so far: 3-constraint . . . 10
18.3.5. Homogeneous 3-m.g.s: forbidden triangles, etc. . . . 12
18.3.6. Homogeneous 2-m.t.s: forbidden triangles . . . 12
18.3.7. The catalog revisited . . . 14

CHAPTER 19. HOMOGENEOUS 3-MULTI-GRAPHS . . . 15
19.1. A catalog . . . 16
19.2. Triangle constraints . . . 19
19.3. The locally degenerate imprimitive case . . . 22
19.3.1. Some general ideas . . . 23
19.3.2. The case $m < \infty$: preliminary analysis . . . 26
19.3.3. Proof that $m = \infty$ . . . 39
19.3.4. Proof that $n = \infty$ . . . 46
19.3.5. The case $m = n = \infty$: structure of $u^p \cap \Gamma_1$ for $u \in \Gamma_2$ . . . 49
19.3.6. The (Γ_1, Γ_2)-type partition . . . 59
19.3.7. Types of $\overset{2}{\sim}$-classes in Γ_1 over pairs in Γ_2 . . . 63
19.3.8. Proposition 19.3.44 . . . 66
19.4. 3-multi-graphs: conclusion . . . 67

CHAPTER 20. IMPRIMITIVE HOMOGENEOUS 2-MULTI-TOURNAMENTS . . . 69
20.1. Introduction . . . 69
20.2. A catalog of homogeneous 2-multi-tournaments . . . 70
20.3. Imprimitive homogeneous 2-multi-tournaments . . . 74
20.3.1. The shuffled case . . . 78
20.3.2. General type, with a finite quotient . . . 84

20.3.3. Infinite quotients and the parity constraint 92
20.3.4. Generic type: 2-restrictions 94
20.3.5. Generic type: structure of u^2 99
20.3.6. Infinite quotient, generic type: conclusion 101

CHAPTER 21. 3-CONSTRAINED HOMOGENEOUS 2-MULTI-TOURNAMENTS 107
21.1. The classification 108
21.2. Existence 111
21.2.1. The exceptional cases: 7–10 111
21.2.2. The structures $\mathbb{S}(n)$ and entries 11 and 12 114
21.3. Completeness: the partially ordered case 118
21.4. Completeness: no partial orders 125
21.4.1. Proof of Proposition 21.4.1 126
21.4.2. Proposition 21.4.2 135
21.4.3. The case of $\mathbb{S}(4)$ 136
21.4.4. Entry #10 141
21.4.5. Entry #8 142
21.4.6. Proof of Proposition 21.4.2 144
21.4.7. Entry 9: Proposition 21.4.3 148
21.5. Completeness: the 3-constrained case 156

CHAPTER 22. HOMOGENEOUS 2-MULTI-TOURNAMENTS: FORBIDDEN TRIANGLES; CONCLUSION 157
22.1. Introduction 157
22.2. The transitive case 158
22.3. No definable p.o.: around $\widetilde{\mathbb{S}(3)}$ 160
22.3.1. Notation: 2-multi-tournaments of order 4 161
22.3.2. The triangle types $L_3(1,1,2)$, $L_3(2,2,1)$, $C_3(1,1,2)$, and $C_3(2,2,1)$ 162
22.3.3. Around $\widetilde{\mathbb{S}(3)}$, continued 170
22.3.4. Reduction to cases (a, c) 172
22.3.5. Case (a) 175
22.3.6. Case (c) 178
22.3.7. Proof of Proposition 22.3.1 180
22.4. Around $\mathbb{S}(4)$ 180
22.4.1. Positive constraints 180
22.4.2. A first dichotomy 182
22.4.3. The bad case 183
22.5. Generalizing Lemma 21.4.25 188
22.6. Around entry #8 201
22.7. Around entry #9 203
22.8. Forbidden triangles: analysis so far 213
22.9. 2-multi-tournaments: conclusion 216

22.9.1. Triangle constraints: an overview 216
22.9.2. Triangle constraints: the catalog 216

APPENDIX B. OPEN PROBLEMS AND SOME RECENT RESULTS 223
B.1. Model theoretic problems 224
B.1.1. Structural Ramsey theory and applications 224
B.1.2. The finite case 228
B.1.3. Model theoretic simplicity, pseudoplanes 234
B.1.4. Ternary languages 234
B.1.5. Finite model prop., finite axiomatizability 237
B.1.6. Homogenizability 238
B.1.7. n-Cardinal properties 239
B.1.8. Partial orders of width n: model companion 240
B.2. Automorphism groups of homogeneous structures 240
B.2.1. The abstract group 241
B.2.2. Dynamical properties 241
B.2.3. Descriptive set theory and Polish groups 246
B.2.4. Regular actions 246
B.3. Algorithmic problems 247
B.3.1. Constraint satisfaction problems 247
B.3.2. Decision problems for model companions 248
B.4. Old chestnuts 251
B.4.1. Locally finite (semifinite) generalized n-gons 251
B.4.2. Stable countably categorical groups 252
B.4.3. Transitivity conditions 252

REFERENCES FOR VOLUME II 253

INDEX 261

LIST OF TABLES

Table 1.1 The Homogeneous Ordered Graphs. 9
Table 1.2 Admissible parameters for $\delta = 3$. 24

Table 2.1 Propositions I–IX. 35
Table 2.2 Amalgamation Strategy. 44

Table 3.1 The Homogeneous Ordered Graphs. 57

Table 4.1 Amalgamation: $A2$. 65

Table 5.1 Propositions I–IX$_n$. 88

Table 11.1 Metrically homogeneous graphs of non-generic type. 180

Table 12.1 Amalgamation Lemmas 12.4.2 to 12.4.9. 206
Table 12.2 Amalgamation, Case II. 208
Table 12.3 Amalgamation, Case III. 208

Table 16.1 Exceptional Bipartite Graphs. 290

ABSTRACT

Volume I. Part I: A complete classification of homogeneous ordered graphs is given: up to a change of language each is either a generically ordered homogeneous graph or tournament, or a generic linear extension of a homogeneous partial order.

Part II: A catalog of the currently known metrically homogeneous graphs is given, with proofs of existence and some evidence for the completeness of the catalog. This includes a reduction of the problem to what may be considered the generic case, and some tools for the analysis of the generic case.

Some related developments are discussed in an appendix.

Volume II. Here the impact of the results of Parts I and II and of related work in Amato, Cherlin, and Macpherson [2021] on the classification of homogeneous structures for a language with two anti-symmetric 2-types or with 3 symmetric 2-types is worked out in detail.

An appendix to Volume II discusses some further advances in related areas, and a wide variety of open problems.

An extensive bibliography of related literature and a quick survey of that literature, organized by topic, is given in Cherlin [2021] (see also http://www.cambridge.org/9781009229692).

The method used in Part I of Volume I is due to Alistair Lachlan. The method used in Part II of Volume I and throughout Volume II is a direct application of Fraïssé's theory of amalgamation classes.

2020 Mathematics subject classifications. Primary 03C15. Secondary 03C10, 03C13, 05C12, 05C55, 05C63, 06F99, 20B27.

Cambridge University Press & Assessment
978-1-009-22969-2 — Homogeneous Ordered Graphs, Metrically Homogeneous Graphs, and Beyond
Gregory Cherlin
Frontmatter
More Information

PREFACE

The notion of homogeneity that concerns us here was first noticed by Urysohn in a metric context, a few days before his tragic death in a swimming accident in 1924. As transcribed by his companion Alexandrov into German for the benefit of Hausdorff, and into French for the benefit of the mathematical public (Hušek [2008], Urysohn [1925]), Urysohn's remark goes roughly as follows.

> U is *homogeneous* in this sense: the finite and congruent sets A, B (lying in U) being arbitrarily chosen, there is an isometric map of U onto itself transforming A into B.[1]

One can trace the notion back further, since it amounts to saying that the Euclidean viewpoint and Felix Klein's coincide on finite configurations. Indeed, this is more or less the point of view taken by Freudenthal in a survey of work on two-point homogeneity (Freudenthal [1956]), which places the issue firmly in the context of ideas of Riemann, Helmholtz, and Lie on the foundations of geometry.

In an algebraic or combinatorial context we speak of *isomorphism* rather than congruence or isometry; and to be precise, we require that *any* given isomorphism extend to an automorphism.

Erdős and Rényi remarked in 1963, with what I take to be some bemusement, that infinite structures with rich automorphism groups appear as natural limits (in a probabilistic sense) of rigid finite structures.

> *Thus there is a striking contrast between finite and infinite graphs: while "almost all" finite graphs are asymmetric, "almost all" infinite graphs are symmetric.* Erdős and Rényi [1963]

Fraïssé's theory of *amalgamation classes* from the 1950s would suggest a more extreme example of the same phenomenon: the rational order is the *Fraïssé limit* of the finite orderings. Fraïssé pointed out the role of the amalgamation

[1] "U est *homogène* en ce sens que, les ensembles finis et congruents A et B (situés dans U) étant quelconques, il existe une représentation isométrique de U sur lui-même transformant A en B." " . . . eine recht starke Homogenitätsbedingung . . . letzterer besteht darin, daß man den ganzen Raum (isometrisch) so auf sich selbst abbilden kann, daß dabei eine beliebige endliche Menge M in eine ebenfalls beliebige, der Menge M kongruente Menge M_1 übergeführt wird."

property as the counterpart at the finite level of the rich automorphism group in the infinite limit, giving one possible answer to the question implicit in the remark of Erdős and Rényi.

The automorphism group of a homogeneous structure carries a natural topology, and with this topology the automorphism group of the ordered set $\mathbb{Q}$ has a remarkable fixed-point property: any continuous action on a compact set has a fixed point. Pestov pointed out in 2002 that this property is equivalent to Ramsey's theorem.

Meanwhile, Nešetřil had observed that under mild conditions, the generalized Ramsey property for a class of finite structures implies that the homogeneous Fraïssé limit exists (Nešetřil [1989], cf. Nešetřil [2005]). And in 2005 Kechris, Pestov, and Todorcevic closed this circle of ideas by showing that for homogeneous structures, the fixed point property for the automorphism group is equivalent to structural Ramsey theory for the finite substructures. Presumably Erdős and Rényi would have found all this very illuminating.

Since then, things have been generally lively, and a good deal of young talent has taken up the subject from a bewildering multiplicity of points of view. We will revisit a little of that in the Appendices to Volumes I and II.

In the 1970s Henson observed that Fraïssé's method gave more examples than one would necessarily want to have,[2] notably an uncountable family of homogeneous directed graphs. In the 1980s Lachlan and Woodrow developed techniques based on Fraïssé's theory, sometimes using the classical Ramsey theorem as well, to *classify* homogeneous structures for restricted languages, showing that all homogeneous graphs and tournaments were known.

In the 1990s I decided to put Henson's examples together with the Lachlan/Woodrow technique in a cage match, uncountably many against classification, and the latter won:[3] the homogeneous directed graphs can be classified, and in fact most of them are the ones Henson originally constructed—that is, the full list contains countably many additional structures (Cherlin [1998]). In the appendix to that work I took a very tentative look at the classification problem for homogenous structures in slightly more complicated languages, having either two anti-symmetric 2-types, or three symmetric 2-types, and later I classified the imprimitive examples of the latter kind.

Volume I. In recent years, I have taken up two more classification problems for homogeneous structures that struck me as of particular interest, which are the subject of the first volume of this monograph.

---

[2]"This book tells me more about penguins than I wanted to know."

[3]This was unexpected. Chapter IV of Cherlin [1998] provides a missing link between the case of homogeneous graphs and the case of homogeneous digraphs which might perhaps have altered my expectations if it had been known beforehand. That chapter is also the main source of the strategy for Part I of this monograph.

The first of these problems is the classification of the *homogeneous ordered graphs*, which was suggested to me in 2012 by Lionel Nguyen Van Thé as a natural problem from a Ramsey theoretic point of view, and potentially the source of interesting new examples. The second problem is the classification of *metrically homogeneous graphs* (equipping a connected graph with its path metric,[4] and requiring metric homogeneity). This problem was noted in passing in Moss [1992] and more explicitly by Cameron, in terms that I find memorable:

> The theory of infinite distance-transitive graphs is open. Not even the countable metrically homogeneous graphs have been determined.
> Cameron [1998]

The complete solution to our first problem, the classification of the homogeneous ordered graphs, will be found in Part I.

The second problem is not yet completely solved. A catalog and a corresponding conjecture as to the classification of the metrically homogeneous graphs will be found in Part II, together with a reduction of the problem to what I call "generic type," and some general structural results which apply in the case of generic type.

In the conjectured classification, the class of 3-*constrained* metrically homogeneous graphs plays a leading role; these are the metrically homogeneous graphs whose finite induced metric substructures can be characterized by forbidden *metric triangles*. I give a completely explicit classification in the 3-constrained case, and somewhat more. The main classification conjecture amounts to a reduction of the classification to the 3-constrained case, in a sense to be discussed below.

Other results in Part II concern the treatment of the *bipartite* case and the case of *infinite diameter*: if all metrically homogeneous non-bipartite graphs of finite diameter are in our catalog, then in fact all metrically homogeneous graphs are known.

Not given in this monograph, but very relevant to it, is the verification of the conjectured classification in the case of diameter 3, which was carried out in Amato, Cherlin, and Macpherson [2021]. That work uses a certain amount of material from the present monograph, but only for the sake of convenience—the general theory supplies some initial reductions which would not be difficult in diameter 3, and that theory also predicts—and explains—the classification found.

Later we realized that one can take the diameter 3 treatment as an indication of an inductive approach to a full proof of the classification conjecture (and then the full content of Chapters 13, 14 of the present work also becomes

[4]Such a structure is called a *distanced graph*, in Moss' terminology. Thus a metrically homogeneous graph is a homogeneous distanced graph.

relevant, beyond the 3-constrained case). This new approach is being actively explored and is discussed further in the Appendix to this volume, in §A.2.1.

That appendix discusses four directions in which there has been substantial recent progress, and which are connected fairly directly to the material of this volume. The reader interested in a broader view of open problems in the area of homogeneity and related parts of model theory will no doubt want to explore the appendix to Volume II as well.

The first two topics dealt with in the appendix to Volume I concern classification problems: namely, the classification of the homogeneous "multi-orders," also called *finite-dimensional permutation structures*, by Braunfeld and Simon, and the ongoing classification project for metrically homogeneous graphs and its relationship to the strategy developed in Amato, Cherlin, and Macpherson [2021], the latter already mentioned.

The other two topics discussed there involve the closer study of the automorphism groups of metrically homogeneous graphs of generic type. One of the goals of a classification project is to uncover interesting, and possibly exotic, examples which are suitable for further study. Certainly the known metrically homogeneous graphs of generic type fall under that heading.

There is now a very rich general theory relating the study of the automorphism groups of homogeneous relational structures to finite combinatorics, following on a breakthrough in the seminal paper by Kechris, Pestov, and Todorcevic [2005]. There is another very interesting line coming from Tent and Ziegler [2013]. Typically these theories reduce the study of automorphism groups of homogeneous structures, viewed as abstract groups, as topological groups, or as permutation groups, to combinatorial problem concerning the associated classes of finite structures.

As far as the theory of automorphism groups of metrically homogeneous graphs is concerned, we confine ourselves in the appendix to Volume I to the combinatorial side of the problem. That is, we discuss the relevant combinatorial properties of finite substructures of metrically homogeneous graphs. These properties relate to *completion procedures* for partial metric spaces embedding in metrically homogeneous graphs.

The desired completion procedures lead to exotic notions of *shortest path completion* in generalized metric spaces which promise to reshape the whole theory of metrically homogeneous graphs conceptually. Among the sources for this material are Aranda et al. [2017], Konečný [2019a], Konečný [2019b]; the first two articles mentioned do not use the language of generalized metric spaces, while the third exploits that point of view enthusiastically, but gives less detail for the case of metrically homogeneous graphs. The published version of Aranda et al. [2017] is Aranda et al. [2021]; this is rather compressed and does not give the most general form of the results.

Finally, we revisit a question of Cameron and Tarzi on splitting the group of "twisted automorphisms" over the group of ordinary automorphisms, taking up the question in the context of metrically homogeneous graphs. The result found in that case provides an interesting, though anecdotal, counterpart to their prior results. What form such results might take in a more general setting, and how broad such a setting should be, remains unclear. But this is an area which invites further investigation.

The first two chapters of Volume I present an overview of the results obtained and the methods used in Parts I and II. That is, the first chapter presents the *results* of Parts I and II in detail, and the second chapter discusses the *methods* used in both parts. Thus readers who have a definite interest in just one of the two topics treated should read these two chapters selectively; depending on their needs or interests, they may then possibly be spared reading the rest of the monograph—but will certainly want to look at the appendix, and very likely at the appendix to Volume II as well.

Volume II. At the end of my earlier monograph on homogeneous directed graphs (Cherlin [1998]) the logical next phase of that project was briefly considered: the classification project for homogenous structures in languages with two pairs of anti-symmetric 2-types, or with three symmetric 2-types. We will refer to structures of the first kind as 2-*multi-tournaments* and to structures of the second kind as 3-*multi-graphs*. With a little computer assistance (specifically, a home computer of 1990s vintage), the 3-constrained 2-multi-tournaments and 3-multi-graphs were found, and were listed in tabular form in Cherlin [1998], with some trivial cases omitted.

The work presented in Part I of the present volume falls wholly within the first (anti-symmetric) setting, while the work in Amato, Cherlin, and Macpherson [2021] falls wholly within the second (symmetric) setting. So it is natural to ask how far this work advances us toward a solution of either of those more general classification problems. The answer is not immediate.

In fact, this question is the subject of Volume II. To be clear, we do not seek a solution to these classification problems in the near term, but rather a road map and an understanding of where the current results actually leave us with respect to the broader problems.

Our experience in Part II of this volume strengthens our sense that the study of 3-constrained cases is an important part of the classification process in its cataloguing phase. In a systematic approach to the classification problem for all homogeneous structure in a fixed, small, binary relational language, one expects to proceed according to the following scheme, which has some theoretical justification (mostly conjectural).

1. Identify the 3-constrained structures.
2. Show that with few exceptions the triangle constraints in a homogeneous structure agree with those in some 3-constrained structure.
3. Classify the homogeneous structures whose triangle constraints do not define a free amalgamation class.

4. Classify the homogeneous structures whose triangle constraints do define a free amalgamation class.

For the last two points, one expects to encounter Henson constraints (suitably understood) as well as some truly exceptional or even sporadic cases.

See §18.2 in Volume II for a fuller discussion of this.

For our purposes point (1) is part of the pre-history, and the present volume (and related work) bears on instances of (3). It seems that something has been skipped!

The missing point (2) turns out to be challenging. The whole of Volume II attempts to address it. That is, we attempt to find all patterns of forbidden triangles in homogeneous 2-multi-tournaments or 3-multi-graphs, with the known classification in the 3-constrained case providing the target for the analysis. At the end, certain recalcitrant cases remain open, which we believe can be eliminated, ideally with some further computer assistance of a more substantial (interactive) type.

Chapter 18 (the first chapter of Volume II) surveys the results obtained on this problem in considerable detail. This chapter serves as an extended introduction to the whole volume.

In Chapter 19 we give a satisfactory treatment of point (2) in the case of homogeneous 3-multi-graphs. We know the possible patterns of triangle constraints, and we find in fact that an unknown homogeneous 3-multi-graph must be infinite and primitive and have triangle constraints compatible with free amalgamation. In other words, we arrive at the start of what should be the generic case, with all of the obvious special cases treated. It is very convenient for our purposes that the imprimitive case has been analyzed separately in earlier work.

Homogeneous 2-multi-tournaments are more recalcitrant. We do not have a prior classification in the imprimitive case, so we next address that point, in Chapter 20. The analysis there is similar to what was done previously in the case of imprimitive homogeneous digraphs.

Next we give the classification of the 3-constrained homogeneous 2-multi-tournaments in detail, without relying on computer computations. Indeed, if we wish to work out the general patterns of constraints on triangles which are compatible with homogeneity, then we need to have such a treatment as a point of departure. So this is the subject of Chapter 21.

Finally in Chapter 22 we arrive at the problem of the determination of the possible triangle constraints for homogeneous 2-multi-tournaments. This is incompletely resolved. Four cases which do not correspond to 3-constrained examples remain elusive. We believe these cases can be excluded with substantial computer assistance, or possibly by a very elaborate line of argument, which we investigated just far enough to see that it has some promise. One would probably do best to combine the approaches: the individual steps of such an analysis are of a type which lends itself to proof by computer, with a lucid proof resulting as output, via a tree search which is very tedious by hand, and has no a priori bound on depth, though in practice the depth seems very small, while the width is exponentially large. These cases are eliminated in the 3-constrained case using amalgamation diagrams of order 6 (factors of order 5). They are the only cases in which the class defined by the constraints allows amalgamation of all diagrams of order at most 5, but is not an amalgamation class.

The numerical point here is that $6 = 4 + 2$ with 4 being the number of 2-types. In the case of the known metrically homogeneous graphs of generic type, of any diameter, amalgamation up to order 5 implies amalgamation. But in general one expects to need amalgamation up to order $r + 2$, where r is the number of 2-types.

An appendix to Volume II considers, very broadly, some open problems in the theory of homogeneous structures, with references to other similar surveys. This may be viewed as a continuation of the appendix to Volume I. This appendix began its life as a short note intended for graduate students, but has evolved since.

We will elaborate further on the contents of Volume II, and on the classification of binary homogeneous structures generally, in a separate introduction to Volume II.

Each volume has its own biliography and index (the latter with few cross references across volumes).

General Remarks. As combinatorialists occasionally remind me, their concern is not so much with the classification of homogeneous structures of a particular kind, but rather with the identification of novel examples.

From that point of view, Part I is a failure (or, if one prefers, it is only due diligence)—there is nothing new to be found in that direction. Part II on the other hand is very successful from that point of view. It contains a rich catalog of new examples and this catalog has itself been studied and to a degree explained by a considerable body of combinatorial work already alluded to.

As we observed, the main feature of this catalog—as such—is the classification of the 3-*constrained* metrically homogeneous graphs of generic type, and the associated *variants with Henson constraints*. In practice the Henson variations do not much complicate matters, and the main combinatorial issues arise already in the study of the 3-constrained structures.

Apparently the assumption of 3-constraint is not in itself useful combinatorially, and what is useful is the reinterpretation of the classification of the 3-constrained metrically homogeneous graphs in terms of a theory of generalized metric spaces with values in a partially ordered semigroup, as a step toward the characterization of the *partial* (i.e., *weak*) substructures of a given 3-constrained metrically homogeneous graph. For the moment, at least, this conceptual description still depends on an explicit classification by direct methods.

So the lesson I would take from this, currently, is that one should focus more on understanding the 3-constrained case in general (possibly under a hypothesis of strong amalgamation and perhaps also primitivity).

One tantalizing feature of recent developments is that the proofs of the amalgamation property under suitable (and quite arcane) conditions given in Chapter 12 have since been reinterpreted as a shortest path completion in a generalized metric space with values in a partially ordered semigroup. We say more about this point of view in the appendix to this volume.

While clarifying, this interpretation in terms of generalized metric spaces does not immediately provide a precise explanation of all of the conditions found, though many of them are required for the construction of the semigroups $D^{\delta}_{M,C}$ described in the appendix, and all of them come in eventually in the treatment of the completion procedure (inevitably). From another point of view, these conditions are the result of applying quantifier elimination to a formula in Presburger arithmetic; this accounts for their general form but does not elucidate their content. Yet another point of view is found in Hubička, Konečný, and Nešetřil [2020c], which may provide a useful heuristic for the classification of 3-constrained homogenous binary structures in other contexts.

* * *

My motivation for the work in Parts I and II is touched on again in the acknowledgments below, and is expanded on in Chapter 1. While I had not expected to return to the broader classification problems considered in Part III after taking up these problems, it now seems very natural to do so—at least, now that the work in Amato, Cherlin, and Macpherson [2021] is complete.

Fans of the Ramsey theoretic approach[5] to the classification of homogeneous structures will be pleased to see it carrying the burden of the argument in Part I. I had thought it might come in also to the general classification project for metrically homogeneous graphs, but it seems not (see the appendix to this volume for current thinking on this point).

Part II introduces a new family of metrically homogeneous graphs. This family was described in Cherlin [2011], but not actually proved to exist there. Here at last the existence proof is given (Chapter 12), along with some useful general theory.

The rest of Part II makes a start on the problem of showing that the catalog of metrically homogeneous graphs found is complete, that is, that the catalog is exhaustive. We carry out reductions of the classification problem to what we call generic type, as well as to the non-bipartite and finite diameter cases (under suitable inductive hypotheses). We also develop, and apply, some general methods of *local analysis*, by which we mean the study of the substructures

[5]It deserves more fans—or practitioners; the method is very powerful. Though it may lend itself more easily to machine-assisted work.

induced on the locus of a 1-type relative to a fixed parameter (basepoint). The immediate prospects for completion of the classification project are discussed in the appendix.

To close, I add a few words about the development of the material presented here.

I began working seriously on the classification of metrically homogeneous graphs in 2006, starting on the side of what I now call *exceptional local type*. The classification for exceptional local type was given in Cherlin [2011] along with the catalog for generic type, including a statement of the classification in the 3-constrained case and a description of the amalgamation procedure. For reasons of space (not to mention a deadline) I did not include the proof that the amalgamation procedure works for the classes in question, nor that these classes exhausted the 3-constrained ones of generic type.

The question addressed in Part I came up in Summer 2012, and it occurred to me soon afterward that the method of Chapter IV of Cherlin [1998] was very likely applicable. The details were worked out in 2013.

At the time, it seemed reasonable to put these two projects together into one monograph, with (as I thought) two Parts, a project initiated in 2015. A third part (now a second volume) made its appearance in 2016. The monograph was submitted in 2017, but from time to time I revisited the third part. That part became both more intelligible but also longer as the results were completed and the proofs somewhat expanded, partly in response to remarks by referees.

Ultimately a projected chapter on homogeneous 2-multi-tournaments became three chapters. In parallel, the treatment of the diameter three case of metrically homogeneous graphs jointly with Amato and Macpherson also expanded, and then led to further developments discussed in the appendix. At present there is good reason to think that we are on the right track for a full proof of the classification of the metrically homogeneous graphs, building on the methods used in diameter 3.

The state of knowledge has shifted considerably over this period. This is most visible in the discussion of open problems, whose perspective is mainly that of 2016, with modest revisions in 2020–21, mainly with respect to our new approach to the proof of the classification of the metrically homogeneous graphs.

The division into volumes, each with its own appendix, is a late decision. There may well be some remarks in the text more reminiscent of 2016 than of 2021, but in any case these would concern matters that remain to be settled.

Princeton, August 2021

Cambridge University Press & Assessment
978-1-009-22969-2 — Homogeneous Ordered Graphs, Metrically Homogeneous Graphs, and Beyond
Gregory Cherlin
Frontmatter
More Information

ACKNOWLEDGMENTS

I am grateful to Lionel Nguyen Van Thé for raising the question treated in Part I of Volume I in summer 2012. This did not strike me immediately as a reasonable question, but my inability to articulate a concrete objection soon forced me to take it seriously. I am also grateful to Miodag Sokić for drawing my attention to the article by Dolinka and Mašulović [2012] in Fall 2013.

With regard to the material of Part II of Volume I, I first encountered the problem of classifying countable metrically homogeneous graphs in Moss [1992], then found the discussion in Cameron [1998] very stimulating, and found evidence in Kechris, Pestov, and Todorcevic [2005] that the problem was natural from several other points of view.

Much of my fascination with the classification of homogeneous structures can be traced back to Alistair Lachlan. I have long enjoyed his understated enthusiasm, his gift for turning examples into theories, and his fearlessness.

In the course of preparing this text, remarks by Dugald Macpherson on the content and Jan Hubička on the literature were very helpful.

Lately Braunfeld, Coulson, Evans, Hubička, Konečný, Nešetřil, and Simon have been among those who have complicated my life by telling me interesting things outside the scope of the monograph that nonetheless deserved mention here and there. A good deal of that found its way into the appendix to Volume I. That list could be longer, but one has to stop somewhere.

I greatly appreciate the careful work done by a number of anonymous referees, as should the reader. (This applies with particular force to Volume II.)

For that matter, I also appreciate the work done by an anonymous referee on Amato, Cherlin, and Macpherson [2021] and the impetus provided to us jointly to reflect further on the path forward with regard to the classification of metrically homogeneous graphs.

I thank Stewart Cherlin for editing the family photographs used as a frontispiece to be suitable for reproduction in black and white.

And I take this opportunity to acknowledge, for once, my lifelong debt to Donald Knuth for the years he took off to make TEX work.

Amélie and Grégoire provided regular and varied distractions; le petit Nicolas came in at the end with music and dance. S. L. Huang was a source of additional entertainment. Christiane and Rufus provided their famous hospitality, and Chantal kept it all together.

Research supported by the Hausdorff Institute for Mathematics, Bonn, Fall 2013 and again in Fall 2018, and by a series of grants from the NSF, including DMS-1362974.