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978-1-009-22969-2 — Homogeneous Ordered Graphs, Metrically Homogeneous Graphs, and
Beyond
Gregory Cherlin
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Chapter 1

RESULTS

1.1. Introduction

We deal with structures in a relational language (often, but not always, a finite language). The problem that concerns us is the classification of the countable homogeneous structures in some natural cases. The two cases discussed here are the following.

1. Countable homogeneous ordered graphs.
2. Countable metrically homogeneous graphs.

Everything will be countable, and it is tempting to drop out that word throughout, but we will try to resist the impulse.

A structure in a relational language is called *homogeneous* (or *ultra-homogeneous*, for emphasis) if every isomorphism between finite substructures is induced by an automorphism. An ordered graph is a graph with a linear ordering; there are no additional constraints.

A *metrically homogeneous* graph is a connected graph with the property that the associated metric space is a homogeneous metric space. The associated metric space has the vertices as points, with distance the minimal path length connecting two vertices.

Metrically homogeneous graphs of diameter 2 are just connected homogeneous graphs. The main examples were given in Henson [1971] using Fraïssé's method, and Lachlan and Woodrow [1980] showed that they were indeed the main examples, by completing the classification of the countable homogeneous graphs, building on a prior classification of the finite ones by Sheehan [1974] and Gardiner [1976] . The Lachlan/Woodrow classification plays a fundamental role in our analysis (see §1.4).

In Part I we will give an explicit, and surprisingly simple, classification of the countable homogeneous ordered graphs. The main ingredient of the proof was found (hidden) in Cherlin [1998, Chapter IV]. In addition, one of the three cases that must be treated was covered in its entirety by Dolinka and Mašulović [2012].

In Part II we will give an explicit, and not particularly simple (but still simple enough), *conjecture* concerning the classification of the countable metrically homogeneous graphs. This conjecture has already been discussed in Cherlin [2011], but we got ahead of ourselves there, stating a good deal more than we proved, for reasons of space and balance—our earlier discussion was intended to be a broad one, though it evolved in a more technical direction. Here we give a full account of the conjecture, which to begin with requires that we prove that the family of new examples described in Cherlin [2011] actually exists, after which we present a number of results which provide some support for the conjecture that our explicitly given list of examples is complete, or very nearly so.

In this introductory chapter we will state our main results on both of these classification problems in detail, and in the next chapter we will indicate the main lines of argument used. Much of the work involved in problems of this kind lies in finding a suitable inductive framework for the proof, and then working out the scaffolding of supporting lemmas on which the argument ultimately depends. This definitely requires a top-down approach, so in the next chapter we will begin at the top.

Before describing the results to be obtained, we review the Fraïssé theory, on which everything done here depends—both the existence of many of the structures in question, as well as their classification.

1.2. Fraïssé limits and amalgamation classes

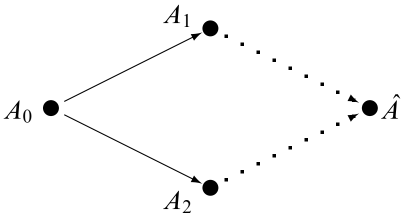
Fraïssé theory plays a fundamental role in classifications of homogeneous structures. We recommend Macpherson [2011] for a survey in the modern spirit. We now give a synopsis of that theory.

Fraïssé observed that a countable homogeneous structure is uniquely determined by the isomorphism types of its finite substructures, and gave an explicit recipe for deriving the structure as a kind of limit of its finite substructures, now called the Fraïssé limit. This limit structure occasionally has a probabilistic interpretation (or more than occasionally, if one forces matters as in Petrov and Vershik [2010], Ackerman, Freer, and Patel [2016], Ackerman et al. [2016]).

The main property required for the construction is the *amalgamation property*: there must be an amalgamation procedure which extends any diagram

$$f_1, f_2 : A_0 \hookrightarrow A_1, A_2$$

involving embeddings of structures in the given class to an amalgam $\hat{A}$, also in the given class, and embeddings $g_1, g_2 : A_1, A_2 \hookrightarrow \hat{A}$, making a commutative diagram.



The fact that the amalgamation property holds for the class of finite structures embedding into a given homogeneous structure is an elementary but useful fact. In both Parts we will use many explicit amalgamation arguments.

In the context of structures of combinatorial type (e.g., purely relational structures), Fraïssé correlates countable homogeneous structures with *amalgamation classes* of *finite* structures. These classes are characterized by the following properties.

- Closure under isomorphism and substructure;
- Joint embedding: any two embed in a third;
- Amalgamation over an arbitrary base;
- Only countably many isomorphism types are represented.

The last condition is superfluous when the relational language is finite.

The structure associated by Fraïssé's construction to an amalgamation class is called its *Fraïssé limit*. We note some examples of Fraïssé limits, some of which have natural probabilistic constructions.

- Finite linear orders: limit, the rational order;
- Finite graphs or tournaments: limit, the random graph (also called the Rado graph) or tournament;
- Finite ordered graphs or ordered tournaments: limit, the randomly ordered random graph or tournament;
- Finite graphs with no n-clique: limit, the generic K_n-free graph (the Henson graph H_n);
- Finite ordered graphs with no ordered n-clique: limit, the generic $\vec{K}_n$-free graph;
- Pairs of linear orders on a single finite set: limit, the generic permutation;
- All partial orders: limit, the generic partial order;
- All finite structures consisting of a partial order and a linear extension of it: limit, the generic linear extension of a generic partial order;
- All integer valued finite metric spaces: limit, the homogeneous universal connected graph with respect to embeddings preserving the path metric (Urysohn graph);
- All integer valued finite metric spaces containing no triangles of *odd* perimeter: limit, the homogeneous universal connected *bipartite* graph with respect to embeddings preserving the path metric.

A technique introduced by Lachlan and Woodrow makes use of a special kind of induction over amalgamation classes to prove classification theorems. This technique was subsequently modified by Lachlan in a way that brings the Ramsey theorem to bear. This method builds on the Fraïssé theory, and provides a systematic approach to the proof of classification theorems for homogeneous structures, notably those whose associated amalgamation class has some form of free amalgamation. Lachlan's version of this method plays an essential role in Part I.

In Cherlin [1988] we gave a streamlined account of Lachlan's original application of his method to the classification of homogeneous tournaments. This provides a detailed introduction to the method in a context less encumbered by exceptional cases.

For the construction of homogeneous ordered structures, the notion of *strong amalgamation* is useful. Strong amalgamation is a sharper version of the amalgamation property, in which we require that every amalgamation problem $A_0 \to A_1, A_2$ have a completion (or "amalgam") in which the images of A_1 and A_2 are disjoint modulo the base. In other words, at the level of the underlying sets, the amalgamation process should be free amalgamation.

Since the amalgamation procedure at the level of the underlying sets is canonical, this condition allows a number of strong amalgamation classes to be combined into one. In particular, the class of finite linear orders is a strong amalgamation class, and hence any other countable homogeneous structure whose associated amalgamation class has strong amalgamation can be equipped with a *generic linear order*, with the result being unique up to isomorphism.

For similar reasons, any countable homogeneous graph whose amalgamation class has strong amalgamation can be *generically oriented* to give a canonical homogeneous directed graph which when symmetrized gives back the original graph.

Strong amalgamation for an amalgamation class is equivalent to a condition on the Fraïssé limit known as triviality of algebraic closure. We will not elaborate here; see Cameron [1990, (2.15), p. 37]. In fact, we will avoid this terminology, and say in this case that the structure has strong amalgamation—an abuse of language, and a confusion of categories.

In another direction, strong amalgamation also permits the probabilistic representations mentioned above (Petrov and Vershik [2010], Ackerman, Freer, and Patel [2016], Ackerman et al. [2016]).

Finite homogeneous structures with more than one element cannot have strong amalgamation; and since a finite linear order is rigid, a finite homogeneous structure cannot be expanded to a homogeneous structure in the language with an additional linear order unless it was rigid to begin with.

Being homogeneous, infinite Fraïssé limits of finite structures of combinatorial type tend to have very rich automorphism groups (with some exceptions when the language is infinite). We have noted the example $(\mathbb{Q}, <)$, the Fraïssé limit of finite linear orders, a limit of rigid structures with a rich automorphism group, as well as the example of the random graph, remarked on by Erdős and Rényi (see the Preface).

Some of the graphs we consider in Part II have infinite diameter, and are considered as metric spaces in the path metric. This takes us out of the comfortable setting of finite relational languages, $\aleph_0$-categorical structures, and oligomorphic permutation groups.

Combinatorially, a metric space is an edge labeled complete graph with some constraints. Model theoretically, a metric space is a relational structure with one relation for each possible distance—and with every pair carrying a unique label. As we do not allow unlabeled edges, this class of structures is not axiomatizable.

Fortunately, the Fraïssé theory does not rely on axiomatizability; but one must pay attention to the requirement that there are only countably many finite structures. In the context of metric spaces, it suffices to restrict the values of the metric to a countable set—this is why Urysohn built his universal complete separable metric space as the completion of a rationally valued metric space. As the metric spaces considered here are integer valued, no difficulties arise from this quarter.

1.3. The classification of countable homogeneous ordered graphs

Homogeneous ordered graphs are homogeneous ordered tournaments, but ordered homogeneous graphs are not ordered homogeneous tournaments.

One of the motivations for studying homogeneous structures of combinatorial type is that they tend to be associated with classes for which a structural Ramsey theorem holds, or in dynamical terms, structures whose automorphism groups are extremely amenable.

More precisely, these structures tend to have metrizable universal minimal flows, which can typically be realized as the space of expansions of the structure by predicates required for the structural Ramsey theorem to hold. Thus the universal minimal flow for the random graph is the space of its linear orders, and the class of finite ordered graphs has the Ramsey property. The existence of a definable order is necessary for the Ramsey property, and frequently a Ramsey theorem is obtained by adjoining a suitable ordering—but not always, and it is unclear what one can say in general about the required expansion.

In the case of homogeneous graphs or homogeneous directed graphs, the natural expansions to Ramsey classes have been found by a systematic study of all cases. The question naturally arises, and was put by Nguyen Van Thé in conversation in the summer of 2012, whether a systematic classification of homogeneous ordered graphs would uncover any "sporadic" cases not familiar to Ramsey theorists. As it turns out—perhaps surprisingly, given past experience—all homogeneous ordered graphs have been noticed already (though not always explicitly given as ordered graphs). The task of Part I is to prove this.

Our present task is merely to state the result to be proved, in a form that suggests the general structure of the proof.

For our purposes, the categories of ordered graphs and ordered tournaments are interchangeable, as we will now explain.

If a structure Γ carries two binary relations R_1 and R_2, then the structure (Γ, R_1, R_2) is equivalent to the structure (Γ, R_1', R_2) where R_1' is the symmetric difference$R_1 \Delta R_2$. If the second relation R_2 gives Γ the structure of a tournament, then (Γ, R_1') will be a graph if and only if (Γ, R_1) is a tournament. This trivial change of language preserves the automorphism group and takes homogeneous structures to homogeneous structures.

In particular, a linear order on Γ defines a tournament, so with R_2 given by $<$, this transformation gives an equivalence between the classification of homogeneous ordered graphs and the classification of homogeneous ordered tournaments.

Similarly, the replacement of a tournament by its reversal, or a graph by its complement, preserves homogeneity. In translating between ordered tournaments and ordered graphs we will actually combine these two transformations, taking a tournament to the graph complement of the symmetric difference with the order (or, in the other direction, reversing the tournament relation obtained). In more explicit terms, this means that the translation between ordered graphs and tournaments used here is the following.

Definition 1.3.1.

(a) If $(\Gamma, \to, <)$ is an ordered tournament, then the associated ordered graph $(\Gamma, —, <)$ has edge relation defined by

$$x — y \iff \to \text{ and } < \text{ agree on } x, y.$$

(b) If $(\Gamma, —, <)$ is an ordered graph, then the associated ordered tournament $(\Gamma, \to, <)$ has arc relation defined by

$$x \to y \iff x — y \text{ and } x < y, \text{ or } x > y \text{ and } x \not— y.$$

For example, the cyclic tournament C_3 of order 3 has two ordered forms, $\vec{C}_3^{+}$ and $\vec{C}_3^{-}$, while as ordered graphs, these are a path and its complement. Compare Figure 1 below, which shows the translation between a pair of ordered oriented cycles, and the corresponding pair consisting of an ordered path on three vertices and its graph complement. When taken as *constraints* (forbidden structures), either as ordered graphs or as ordered tournaments, they will play an important role in the case division to be presented below. At early stages of the analysis it will be useful to view them as ordered tournaments, and later it will be more convenient to view them as ordered graphs.

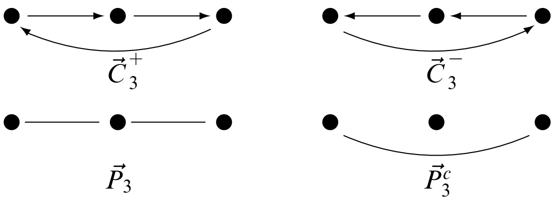


Figure 1. Ordered Tournaments and Ordered Graphs.

So for our purposes, homogeneous ordered tournaments may be viewed as homogeneous ordered graphs, and conversely. But in general, ordered homogeneous tournaments are not ordered homogeneous graphs! We next address this point.

A natural way to produce examples of homogeneous ordered graphs is to begin with a homogeneous graph or a homogeneous tournament with strong amalgamation, and then generically add a linear ordering; in terms of Fraïssé theory, we replace a given strong amalgamation class $\mathcal{A}$ by all ordered forms of the structures in $\mathcal{A}$.

The homogeneous ordered graphs arising from homogeneous graphs or homogeneous tournaments in this way are not the same, though there is a little overlap: generically ordering a random tournament or a random graph gives the same structure, up to a change of language.

In addition to these two sources of examples, there is a third and less obvious source of homogeneous ordered graphs. If Γ is a homogeneous partial order, then we may take a generic linear extension of the partial order—again, under the hypothesis that the corresponding amalgamation class has strong amalgamation. The resulting structure will be homogeneous, and may be viewed as an ordered graph. The graph structure is obtained by symmetrizing the partial order, in other words the edge relation is comparability in the ordering; conversely, the partial order is the intersection of the linear order with this edge relation.

We may also pass to the complementary graph. For examples of the first two kinds, this corresponds to taking the complement of the original graph, or the reversal of the original tournament; so this gives nothing new. But if we take complements of examples of the third kind, we get a fourth kind.

The classification theorem states that we have now described all homogeneous ordered graphs.

Theorem 1.3.2 (Classification of Homogeneous Ordered Graphs). *The countable homogeneous ordered graphs are the following, up to a change of language.*

(*a*) $(\Gamma, \prec, <)$ *with* $(\Gamma, \prec)$ *a countable homogeneous partial order with strong amalgamation and* $<$ *a generic linear extension of* $\prec$;
(*b*) $(\Gamma, \to, <)$ *with* $(\Gamma, \to)$ *a countable homogeneous tournament with strong amalgamation and* $<$ *a generic linear ordering of it*;
(*c*) $(\Gamma, —, <)$ *with* $(\Gamma, —)$ *a countable homogeneous graph with strong amalgamation and* $<$ *a generic linear ordering of it*.

We can easily make this statement more explicit, since there are known classification results for countable homogeneous partial orders, graphs, and tournaments. Indeed, we *must* make the statement more explicit in order to prove it, since our method of proof involves an exhaustive treatment of all possibilities.

We then arrive at the catalog shown below as Table 1.1, which is organized according to the natural order of proof, in terms of the complexity of the minimal forbidden structures for the structure.

The *Type* label refers to the four types of structures involved: *EPO* and EPO^c stand for linear extensions of homogeneous partial orders, and their complementary graphs (once these structures are coded as graphs); *LT* and *LG* stand for generic linear extensions of homogeneous tournaments or graphs, with the ambiguous case noted. Subscripts on *EPO* refer to the particular partial order involved—details are given in the text, in §3.2. The notations for the minimal constraints in the *Forbidden* column are also described in that section. From the ordered graph theoretic point of view, the symbol $A \perp B$ represents a disjoint sum, with A preceding B.

We remark that our notational conventions thoroughly mix notations for ordered tournaments ($\vec{C}_3^{\pm}$) with notation for ordered graphs ($\vec{I}_1 \perp \vec{K}_2$, $[\vec{I}_1, \vec{I}_2]$, etc.), In particular, even when we adopt the point of view of ordered tournaments, the illustrations are more legible when presented as ordered graphs, because fewer edges are required (the order, left-to-right, is implicit).

One remarkable byproduct of our analysis is that the classification of countable homogeneous tournaments is closely related to the classification of countable homogeneous graphs. The main point in the classification of homogeneous tournaments is the fact that a countable homogeneous tournament which is not a local order is the random tournament. If we restrict our attention to homogeneous structures corresponding to strong amalgamation classes, this may be proved as follows: add a linear order generically and then see that we fall under one of the cases IIIB, IIIC in Group III. By inspection, one may show that the only one of these with a homogeneous tournament underlying it is the generically ordered random graph, and that the relevant tournament is the random tournament.

We will describe the proof of this classification theorem in more detail in §2.1. But everything in Group I and II is either covered by Dolinka and Mašulović [2012], or is reducible to it by passing to the complement. The odd-looking

I	*Graphs omitting $\vec{I}_2$ or $\vec{K}_2$*		
Label	*Structure*	*Forbidden*	*Type*
I.1	$\lvert\Gamma\rvert = 1$	$\vec{K}_2, \vec{I}_2$	Triv
I.2	$(\mathbb{Q}, <) = \vec{K}_\infty$	$\vec{I}_2$	EPO_0, LT, LG
I.2^c	$(\mathbb{Q}, >) = \vec{I}_\infty$	$\vec{K}_2$	EPO_0, LT, LG
II	*Graphs containing $\vec{I}_2$ and $\vec{K}_2$, but not both $\vec{C}_3^{\,+}$ and $\vec{C}_3^{\,-}$*		
Label	*Structure*	*Forbidden*	*Type*
II.1	$\mathbb{Q}[\mathbb{Q}^{\mathrm{op}}] = \vec{K}_\infty[\vec{I}_\infty]$	$\vec{C}_3^{\,+}, \vec{I}_1 \perp \vec{K}_2$, and $\vec{K}_2 \perp \vec{I}_1, \vec{C}_3^{\,-}$	$\mathrm{EPO}_\perp$
II.2	Generic permutation	$\vec{C}_3^{\,+}, \vec{C}_3^{\,-}$	LT
II.3$_n$	$\vec{I}_n * \vec{K}_\infty$ dense, with each class dense $(n \cdot \mathbb{Q}$, shuffled); $n \geq 2$	$\vec{C}_3^{\,+}$, $[\vec{I}_1, \vec{I}_2]$, $[\vec{I}_2, \vec{I}_1]$ and $\vec{I}_{n+1}$ (if $n < \infty$) and $\vec{K}_{n+1}$ (if $n < \infty$)	$\mathrm{EPO}_\to$
II.4	$\vec{\mathcal{P}}$=Generic linear extension of generic p.o.	$\vec{C}_3^{\,+}$	EPO_g
II.1^c	$\mathbb{Q}^{\mathrm{op}}[\mathbb{Q}] = \vec{I}_\infty[\vec{K}_\infty]$	$\vec{C}_3^{\,+}$,$[\vec{I}_1, \vec{I}_2]$, $[\vec{I}_2, \vec{I}_1]$, $\vec{C}_3^{\,-}$	$\mathrm{EPO}^c_\perp$
II.3^c_n	$\vec{K}_n * \vec{I}_\infty$ dense, with each class dense	$\vec{C}_3^{\,-}$, $\vec{I}_1 \perp \vec{K}_2$, $\vec{K}_2 \perp \vec{I}_1$	$\mathrm{EPO}^c_\to$
II.4^c	Reversal (complement) of II.4	$\vec{C}_3^{\,-}$	EPO^c_g
III	*Graphs containing both $\vec{C}_3^{\,+}$ and $\vec{C}_3^{\,-}$*		
Label	*Structure*	*Forbidden*	*Type*
IIIA	$\vec{\mathbb{S}}$ = Generically ordered $\mathbb{S}$	$[I_1, C_3]$ and $[C_3, I_1]$ (all ordered forms)	LT
IIIB$_n$	$\vec{H}_n$ = Generically ordered Henson graph $(n < \infty)$	$\vec{K}_{n+1}$	LG
IIIB^c_n	$\vec{H}^c_n$	$\vec{I}_{n+1}$	LG
IIIC	$\vec{\Gamma}_\infty$ = Generically ordered random graph	none	LT, LG

TABLE 1.1. The Homogeneous Ordered Graphs.

entry at (II.2), the generic permutation, is most naturally thought of as the generic linear extension of the tournament $(\mathbb{Q}, <)$, and is classified as type *LT*. But it happens to be a linear extension of a non-homogeneous partial order (namely, the intersection of the two orders) and as such is picked up by Dolinka and Mašulović [2012].

So the main points are how to organize matters so as to take advantage of the case covered by Dolinka and Mašulović [2012]—via complementation this gives two cases—and how to deal with and to distinguish the two kinds of examples that both fall under Group III. Indeed, the four cases in the catalog correspond to distinct portions of the analysis, with no overlap between them.

We will return to the discussion of the methods used to classify the homogeneous ordered graphs in §2.1. In the remainder of this chapter we will describe the results to be obtained in Part II on the classification of countable metrically homogeneous graphs. This requires a lengthier presentation, spanning four sections.

1.4. Countable metrically homogeneous graphs of known type: a catalog and a conjecture

The main goals of Part II are to present a conjecture on the classification of the countable *metrically homogeneous graphs* and some supporting evidence. In this section we discuss the conjecture, and in the following sections, the supporting evidence. Then in the final section of this chapter we will describe a body of results of a more general character on which our more concrete results depend, which fall under the heading of *local analysis*.

The main conjecture may be stated as follows, using some notation which requires elucidation.

CONJECTURE 1 (Metric Homogeneity Classification Conjecture).
The countable metrically homogeneous graphs are the following.

1. *In diameter* $\delta \leq 2$: *the connected homogeneous graphs, classified by Lachlan and Woodrow* [1980]; *Fact* 1.4.2.
2. *In diameter* $\delta \geq 3$:
   (*a*) *The finite ones, classified by Cameron Cameron* [1980]; *Fact* 1.4.5.
   (*b*) *Macpherson's regular tree-like graphs* $T_{m,n}$ *with* $m, n \leq \infty$, $m, n \geq 2$; §1.4.3.
   (*c*) *The Fraïssé limits of amalgamation classes of the form*

$$\mathcal{A}_3 \cap \mathcal{A}_H$$

   with $\mathcal{A}_3$ 3-*constrained and* $\mathcal{A}_H$ *of Henson type or antipodal Henson type*; §§2.2, 1.4.5.

This formulation is compact, but also unintelligible without further explanation. We will make the conjecture completely explicit on a line-by-line basis. It will turn out that the meaning of clauses (1) and (2c) is essential throughout Part II, while the meaning of clauses (2a, b) is entirely irrelevant.

The Lachlan/Woodrow classification remains important throughout, because we can use it to formulate a simple notion of metrically homogeneous