Homogeneous Ordered Graphs, Metrically Homogeneous Graphs, and Beyond

Volume II: 3-Multi-graphs and 2-Multi-tournaments

This is the second of two volumes by Professor Cherlin presenting the state of the art in the classification of homogeneous structures in binary languages and related problems in the intersection of model theory and combinatorics. Researchers and graduate students in the area will find in these volumes many far-reaching results and interesting new research directions to pursue.

This volume continues the analysis of the first volume to 3-multi-graphs and 3-multi-tournaments, expansions of graphs and tournaments by the addition of a further binary relation. The opening chapter provides an overview of the volume, outlining the relevant results and conjectures. The author applies and extends the results of Volume I to obtain a detailed catalogue of such structures and a second classification conjecture. The book ends with an appendix exploring recent advances and open problems in the theory of homogeneous structures and related subjects.

Gregory Cherlin is Distinguished Professor Emeritus at Rutgers University. He has worked on applications of model theory to algebra and combinatorics for half a century, and has published four books and over 100 articles on model theory and its applications.

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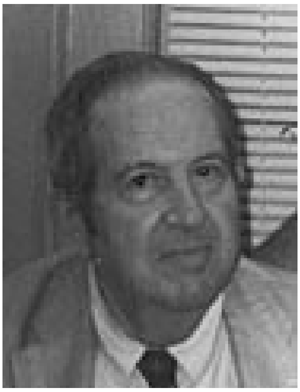
In memory of George Cherlin, 1924–1992.
Enthralled by the beauty of mathematics,
ever mindful of its power for good or evil.





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ABSTRACT

Volume I. Part I: A complete classification of homogeneous ordered graphs is given: up to a change of language each is either a generically ordered homogeneous graph or tournament, or a generic linear extension of a homogeneous partial order.

Part II: A catalog of the currently known metrically homogeneous graphs is given, with proofs of existence and some evidence for the completeness of the catalog. This includes a reduction of the problem to what may be considered the generic case, and some tools for the analysis of the generic case.

Some related developments are discussed in an appendix.

Volume II. Here the impact of the results of Parts I and II and of related work in Amato, Cherlin, and Macpherson [2021] on the classification of homogeneous structures for a language with two anti-symmetric 2-types or with 3 symmetric 2-types is worked out in detail.

An appendix to Volume II discusses some further advances in related areas, and a wide variety of open problems.

An extensive bibliography of related literature and a quick survey of that literature, organized by topic, is given in Cherlin [2021] (see also http://www.cambridge.org/9781009229692).

The method used in Part I of Volume I is due to Alistair Lachlan. The method used in Part II of Volume I and throughout Volume II is a direct application of Fraïssé's theory of amalgamation classes.

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PREFACE TO VOLUME II

A comprehensive introduction to both volumes of this work is given in Volume I. We give a briefer introduction to Volume II here.

In Volume I we considered two classification problems, the classification of the homogeneous ordered graphs and the classification of metrically homogeneous graphs, in Parts I and II respectively. We gave a full solution to the first problem and provided an explicit catalog and conjecture, a body of relevant theory, and some applications in Part II. Additional work in Amato, Cherlin, and Macpherson [2021] confirms the conjecture in diameter 3 and further discussion of the prospects for the general case is found in the appendix to Volume I.

As we explained in the preface to Volume I, the work in that volume together with the work in diameter 3 casts some light on the classification of the homogeneous structures with two pairs of anti-symmetric 2-types or with 3 symmetric 2-types, which we call 2-*multi-tournaments* or 3-*multi-graphs*, respectively. The task of the present volume is to see where exactly these results leave us, with respect to the broader problems. Clarifying this point requires us to undertake some further substantial explorations, some of which might be considered logically prior to the material of Volume I, from the point of view of a systematic study.

Recall that we work throughout with amalgamation classes in accordance with the Fraïssé theory, which amounts to characterizing homogeneous structures by their forbidden substructures. We call a homogeneous structure 3-*constrained* if the minimal forbidden substructures have order at most 3. In dealing with explicit classification problems for binary relational languages, leaving aside the specialized techniques useful for some particular classes (such as finite structures), the general approach to making a catalog (potential classification) of homogeneous structures and investigating its completeness is the following.

1. Classify the 3-constrained homogeneous structures of the desired type.
2. Show that, with few exceptions, the triangle constraints in any homogenous structure of the desired type agree with the triangle constraints in some 3-constrained structure.

3. In all cases for which the triangle constraints are inconsistent with free amalgamation, classify the resulting structures, which are considered to be of exceptional type from the point of view of the general problem.
4. In the remaining case, show, again with few exceptions, that the remaining homogeneous structures are associated with free amalgamation classes.

If this last step ever breaks down—as well it might—we should see something distinctly new appearing. However our focus now is on the first three steps.

The first step was actually carried out in Cherlin [1998], with computer assistance. In the case of 2-multi-tournaments we have to redo this by an explicit argument in order to prepare the way properly for step (2).

One notices that the work carried out in Part I of Volume I and in Amato, Cherlin, and Macpherson [2021] bears specifically on instances of step (3) in each of the two contexts, and that there are indeed some other points that have to be addressed in order to draw definite conclusions as to where the broader classification stands.

This explains why we felt the present volume was necessary. We had thought it would be a supplementary third part to the two parts of Volume I, but discovered that there was a great deal of unfinished business still to be taken care of to bring this to a satisfactory state.

What was achieved, and how, will be described in considerable detail in Chapter 18 (we continue the numbering of chapters, and of open problems, from Volume I). Here we give an overview.

The results for homogeneous 3-multi-graphs are quite satisfactory. They occupy Chapter 19, and are summarized in §19.4. We find that an unknown homogeneous 3-multigraph must be infinite and primitive, and the only forbidden triangles are monochromatic (mainly due to the classification theorem of Amato, Cherlin, and Macpherson [2021], as well as prior classifications in finite and imprimitive cases). These constraints must also be compatible with free amalgamation. With this as a point of departure one may expect, or in any case hope, to find that what remains are in fact free amalgamation classes. At that stage, the powerful classical methods of Lachlan and Woodrow may well become relevant.

On the other hand, in the case of 2-multi-tournaments, we find we have a good deal more work to do. At the end of our analysis four very recalcitrant cases remain, which do not correspond to 3-constrained examples and therefore should, conjecturally, be eliminated.

To begin with, there is no prior classification of homogeneous 2-multi-tournaments in the imprimitive case, so we supply one in Chapter 20. Then we give a proof of the classification of the 3-constrained homogeneous 2-multi-tournaments (Chapter 21). This is needed because we require more than the resulting list of examples—we need to understand why there are no other examples under the strong assumption of 3-constraint. The next challenge is to reach the same conclusion concerning triangle constraints under the assumption of homogeneity alone.

Here a lengthy analysis of the possible patterns of forbidden triangles leads to the identification of four potentially exceptional cases, not corresponding to 3-constrained structures, but not yet ruled out. See Proposition 22.1.1 and §22.8, as well as §22.9.1.

If one uses explicit amalgamation arguments as we do here, one finds that the exceptional cases are the ones for which every amalgamation diagram on five points can be completed, but not every such diagram on six points can be. In all other cases one can use a series of amalgamation diagrams of order at most five to carry out the analysis; these have factors of order four which while compatible with the triangle constraints may or may not embed in the given structure, but in all cases one arrives eventually at a contradiction.

Leaving aside these four delicate cases—that is, moving on to step (3) of our plan—and returning to the patterns of forbidden triangles which *do correspond* to a 3-constrained class without free amalgamation, and which are *not already covered by Part I*: these are the seven examples shown in Table 21.1, under Groups III and IV, labeled 6–12 (as part of a longer list given earlier).

Now Proposition 22.4.1 gives the classification of the homogeneous 2-multi-tournaments falling under case #12 in Table 21.1. So in order to reach what one may consider the generic case of the classification problem for homogeneous 2-multi-tournaments, one must not only eliminate the four exotic patterns of forbidden triangles already identified, but complete six additional classification problems defined by patterns of forbidden triangles which correspond to 3-constrained homogeneous 2-multi-tournaments. We have not assessed the complexity of these problems. The ones treated in Part I of Volume I were complex, the one just mentioned as case #12 is relatively straightforward, and the rest deserve further study.

Princeton, August 2021

ACKNOWLEDGMENTS

I am grateful to Lionel Nguyen Van Thé for raising the question treated in Part I of Volume I in summer 2012. This did not strike me immediately as a reasonable question, but my inability to articulate a concrete objection soon forced me to take it seriously. I am also grateful to Miodag Sokić for drawing my attention to the article by Dolinka and Mašulović [2012] in Fall 2013.

With regard to the material of Part II of Volume I, I first encountered the problem of classifying countable metrically homogeneous graphs in Moss [1992], then found the discussion in Cameron [1998] very stimulating, and found evidence in Kechris, Pestov, and Todorcevic [2005] that the problem was natural from several other points of view.

Much of my fascination with the classification of homogeneous structures can be traced back to Alistair Lachlan. I have long enjoyed his understated enthusiasm, his gift for turning examples into theories, and his fearlessness.

In the course of preparing this text, remarks by Dugald Macpherson on the content and Jan Hubička on the literature were very helpful.

Lately Braunfeld, Coulson, Evans, Hubička, Konečný, Nešetřil, and Simon have been among those who have complicated my life by telling me interesting things outside the scope of the monograph that nonetheless deserved mention here and there. A good deal of that found its way into the appendix to Volume I. That list could be longer, but one has to stop somewhere.

I greatly appreciate the careful work done by a number of anonymous referees, as should the reader. (This applies with particular force to Volume II.)

For that matter, I also appreciate the work done by an anonymous referee on Amato, Cherlin, and Macpherson [2021] and the impetus provided to us jointly to reflect further on the path forward with regard to the classification of metrically homogeneous graphs.

I thank Stewart Cherlin for editing the family photographs used as a frontispiece to be suitable for reproduction in black and white.

And I take this opportunity to acknowledge, for once, my lifelong debt to Donald Knuth for the years he took off to make TEX work.

Amélie and Grégoire provided regular and varied distractions; le petit Nicolas came in at the end with music and dance. S. L. Huang was a source of additional entertainment. Christiane and Rufus provided their famous hospitality, and Chantal kept it all together.

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