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Gregory Cherlin
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Chapter 18

CLASSIFICATION PROBLEMS FOR SMALL BINARY LANGUAGES

In this volume we will consider the material of Parts I and II, together with the result of Amato, Cherlin, and Macpherson [2021], in a substantially broader context, which had been touched on in the Appendix to Cherlin [1998]: namely, lists of 3-constrained structures for small languages were given there, leaving aside the straightforward cases of imprimitive structures or structures with free amalgamation.

I had assumed at first that the consideration of these classification problems would be one of many open problems to be passed in review, in the manner of the appendix to Volume I or the present volume. It quickly became clear that due diligence requires a good deal more in these cases. So clearing up this point, as far as I was able, has become the subject of this volume.

The present Chapter provides an overview of the volume, stating the relevant facts and conjectures, and some associated concrete open questions. The following chapters then prove the various points that have not been dealt with previously.

It is reasonable to ask whether the class of homogeneous structures for a finite relational language is classifiable in any sense at all in general: a reasonable interpretation of this question was given by Lachlan, and we review his suggestion in §18.1, along with another related problem.

Descending (abruptly) to the level of structures homogeneous for a binary relational language, one may formulate the question more narrowly, asking in effect whether there are any examples fundamentally different from those we have already seen. We take a stab at formulating this question precisely in §18.2. In the symmetric case (that is, all 2-types are symmetric) one can go further, as discussed already in the appendix fo Volume I.

After these general considerations we return to the main line of the present monograph. From the point of view of Cherlin [1998], Volume I of the present monograph deals with particular fragments of natural classification problems of a type similar to the one treated in that memoir. Namely, after the classification of homogeneous directed graphs, the next two classification problems to consider in the same spirit would be either the classification of

2-multi-tournaments or 3-multi-graphs. By this we mean tournaments with a coloring of the arcs by two colors, or complete graphs with a coloring of the edges by 3 colors, respectively. In model theoretic terms, we would refer to the class of structures with two pairs of anti-symmetric atomic 2-types, or with three symmetric 2-types, respectively. Our choice of terminology here is intended to avoid major clashes with a number of similarly named notions found in the literature. In particular, the term 3-*graph* has been used in the homogeneous context both for graphs with edge colorings and for 3-hypergraphs, while 2-*graph* conventionally has yet another meaning.

As we will discuss, the material of Part I falls within the classification problem for 2-multi-tournaments, and the material of Part II, when one sets $\delta = 3$, falls within the classification problem for 3-multi-graphs; that special case is resolved in Amato, Cherlin, and Macpherson [2021]. Therefore it is natural to take up those two broader problems here, to take stock of the situation, and to continue the analysis somewhat further in each case.

The first step in any classification problem is to make a catalog, or, as Cameron puts it, a "census" of the known examples, which includes making known any that ought to be known at the outset. From the point of view of the classification problems for homogeneous 2-multi-tournaments or 2-multi-graphs, everything in Part I, or in Part II with $\delta = 3$, falls under the heading of census-taking, without however exhausting the subject. Before leaving the subject, we bring the census to a natural stopping point (but not completion). This is discussed in §18.3, and then the work is carried out in Chapter 19 for 3-multi-graphs and in Chapters 20–22 for 2-multi-graphs.

There are many striking conjectures on homogeneous structures for a finite relational language. We have little immediate prospect of settling any of them. But the results of classification projects have provided some examples which have turned out to be helpful in stimulating the development of the relevant techniques.

Notably, since the completion of the first draft of this monograph, a number of lines of development have contributed to a better understanding of the known binary homogeneous structures. These have been discussed in the Appendix to Volume I but we recall them here.

The first line is the application of the theory of ample generics and structural Ramsey theory via a close study of the "partial" metrically homogeneous graphs. This involves work of Aranda Lopez, Bradley-Williams, Coulson, Evans, Hng, Hubička, Karamanlis, Kompatscher, Konečný, Nešetřil, and Pawliuk. A second line is the study of stationary independence relations in the sense of Tent and Ziegler, and involves work of Li.

These two lines, and a good deal of other prior work, lead to a very broad concept of generalized metric space with values in a finite partially ordered commutative semigroup which can account for the "exotic" 3-constrained examples in the appendix to Cherlin [1998] and also helps in understanding the properties of the 3-constrained metrically homogeneous graphs of generic type. This setting includes the generalized ultrametric spaces used by Braunfeld in studying Cameron's problem on structures with finitely many linear orders, and Conant's generalized metric spaces. More may be found in the Appendix to Volume I.

This development reinforces the impression that to date, the stock of examples one has is not all that varied. It remains to be seen whether this way of viewing these examples is limited to the case of symmetric binary languages or can be extended to cover anti-symmetric cases as well (but here one could use additional examples as test cases).

One should also take note of a radically different approach due to Pierre Simon which uses ideas of geometric neostability theory. While not yet very general, it yields results that seem inaccessible by more direct methods when it applies.

18.1. Lachlan's classification problem

Lachlan posed "the" classification problem for homogeneous structures in finite relational languages in its broadest formulation.

Problem 9 (Lachlan). *Is the following problem decidable*?

> *Given two finite collections of structures $\mathcal{A}_+, \mathcal{A}_-$ in a finite relational language, determine whether there is an amalgamation class containing $\mathcal{A}_+$ and disjoint from $\mathcal{A}_-$.*

In other words, is the classification of homogeneous structures in finite relational languages an art or a science?

One path to decidability might be as follows: one can check easily whether a particular amalgamation problem has a solution. If one could bound the sizes of the amalgamation problems that need to be checked, this would suffice.

For a specified relational language, one may dispose of the problem by any sort of reasonably explicit full classification. For example, there are uncountably many homogeneous directed graphs, but the explicit classification given in Cherlin [1998] passes Lachlan's test: it immediately gives a decision procedure for Lachlan's problem in that setting.

A variant of Lachlan's problem arises as a practical matter in the course of explicit classifications.

Problem 10. *Is the following problem decidable*?

> *Given a finite collection of structures $\mathcal{A}_-$ in a finite relational language, determine whether the class of finite structures containing no isomorphic copy of a structure in $\mathcal{A}_-$ is an amalgamation class.*

In binary languages this problem is decidable, as it suffices to consider amalgamation problems with just two points outside the base, and one can list the possible obstructions to completion in this case. This is helpful: several explicit amalgamation arguments in each of the first two parts of the present monograph were found by following this line of analysis.[1]

Other variants of Lachlan's problem are of combinatorial interest, notably the following.

PROBLEM 11. *Given $\mathcal{A}_+$, $\mathcal{A}_-$ as above, determine whether the number of amalgamation classes containing $\mathcal{A}_+$ and disjoint from $\mathcal{A}_-$ is*

(*a*) *finite;*
(*b*) *countable.*

The decidabilty of version (*b*) of the problem is open even in the case of homogeneous directed graphs, where it becomes equivalent to an instance of the following (for the class of tournaments).

PROBLEM 12 (WQO Problem). *Let $\mathcal{A}_-$ be a finite set of finite structures in a fixed finite relational language, and let $\mathcal{Q}$ be the quasi-ordered class of finite structures not containing any structure in $\mathcal{A}_-$, ordered by isomorphic embedding. Determine whether the class $\mathcal{Q}$ contains an infinite antichain (equivalently, whether the class $\mathcal{Q}$ is "well quasi-ordered"—wqo).*

I have discussed this problem at length in Cherlin [2011a], so I will not elaborate here. It is also well known in the context of permutation pattern classes, where the wqo property holds in a number of cases of interest, and has strong implications for other properties of interest.

In practice this problem leads to an attempt to classify the "minimal antichains" in the given class of structures. In the context of permutation patterns a rich set of such antichains has been identified; richer than in the case of tournaments.

This suggests the following problem.

PROBLEM 13. *Give an encoding of permutations as tournaments that allows the minimal antichains of permutation patterns to be interpreted as minimal antichains of tournaments.*

This seems a little too hard as stated, and probably one should settle for an encoding which works for the known minimal antichains. (Preserving minimality seems challenging.)

An example of an undecidable problem with some connections to homogeneity (specifically, the existence of universal graphs with specified constraints) is given in Cherlin [2011b]. One might take this as a hint of a negative solution to Lachlan's problem.

The problem of decidability of j.e.p. for permutation pattern classes determined by finitely many constraints is also open. On the other hand the

[1] A recent preprint connects this problem to the study of context-free languages. This is a striking development which presumably is worth looking at model theoretically. See Bodirsky, Knäuer, and Rydval [2021].

homogeneous permutations are known; there are finitely many of them and Lachlan's problem trivializes in that context.

18.2. Standard binary homogeneous structures

The supply of known binary homogeneous structures, while extensive, is not very varied. At some point, after becoming convinced that the conjectured classification of the metrically homogeneous graphs put forward in Part II is reasonable, I began to wonder whether something very similar occurs in general. Phrasing this thought explicitly produces a number of concrete classification problems and a systematic way to look for "natural" examples very broadly.

We define *standard* binary homogeneous structures as follows, and then ask whether the classification of binary homogeneous structures reduces, in a very weak sense, to the standard case.

DEFINITION 18.2.1. Let $\mathcal{A}$ be an amalgamation class of finite binary structures.

1. An *amalgamation strategy* γ for $\mathcal{A}$ is a function on 2-point amalgamation problems P over $\mathcal{A}$ such that $\gamma(P)$ supplies a 2-type which can be used to complete the diagram.

2. A *Henson constraint* relative to γ is a finite binary structure whose 2-types lie outside the range of γ.

3. A *standard* binary homogeneous structure Γ is one whose associated amalgamation class $\mathcal{A}$ has the form

$$\mathcal{A} = \mathcal{A}_3 \cap \mathcal{A}_{\gamma,H}$$

where $\mathcal{A}_3$ is a 3-constrained amalgamation class, γ is an amalgamation strategy for $\mathcal{A}_3$, and $\mathcal{A}_{\gamma,H}$ is the class defined by a set of γ-Henson constraints.

The conjectured classification of metrically homogeneous graphs of generic type from Part II of the previous volume amounts, abstractly, to the conjecture that these structures are standard. Much of Part II is then devoted to figuring out what this means concretely. In this case, γ-Henson usually means $(1,\delta)$-Henson.

PROBLEM 14. *Is there a binary homogeneous structure Γ which satisfies neither of the following conditions?*

(a) Γ *is standard.*

(b) *There is $a \in \Gamma$ and a non-trivial 2-type p for which*

$$a^p = \{x \mid \mathrm{tp}(a,x) = p\}$$

realizes fewer triangle types than Γ.

As all imprimitive structures fall under case (b), this is a problem about primitive binary homogeneous structures. One would much prefer to have all non-trivial 1-types over a realize fewer triangles, and perhaps in the primitive case this is reasonable. Clause (b) allows for cases like the generic local order which are primitive but not standard.

There is a long-standing conjecture of a considerably more explicit form in the finite case (as we describe below, a proof has now been announced).

CONJECTURE 1. *Every finite primitive binary structure is of one of the following forms.*

(*a*) Sym_n *acting naturally*;
(*b*) *A cyclic group of prime order acting regularly*;
(*c*) *An anisotropic affine orthogonal group over a finite field acting naturally* (*necessarily of dimension at most* 2).

Note that even finite cliques are standard in our sense (barely): there is an amalgamation strategy which uses no non-trivial 2-type, so any constraint may be considered a Henson constraint. This stretches the notion of free amalgamation and one may prefer to add finite cliques as an explicit exceptional case.

One may check that the affine anisotropic orthogonal groups in dimension 2 also satisfy the stated conditions. In odd characteristic after fixing a point the locus of a 1-type becomes imprimitive with respect to the relation $y = \pm x$. In even characteristic with the order q of the field at least 4 one can check that some 2-type is omitted in one of the induced 1-types. For $q = 2$ we have a finite clique.

Wiscons reduced the proof of Conjecture 1 to the case of primitive actions of almost simple groups in Wiscons [2016], and a systematic attack on the almost simple case by Dalla Volta, Gill, Hunt, Liebeck, and Spiga completed the treatment of all such cases, about the time this book was submitted for publication. (See Appendix B.)

The companion to Problem 14 is the following.

PROBLEM 15. *Classify the* 3*-constrained binary homogeneous structures explicitly.*

The problem is really to classify the standard ones, but dealing with the 3-constrained case would be the heart of the matter.

In practice, Problem 15 has been part of most censuses which have been undertaken, but has tended to involve isolated examples or the imprimitive case. At present this problem is emerging as a center of attention, accompanied by the question to what extent the solution fits into the framework of generalized metric spaces when the 2-types are all symmetric.

Once one has a grip on the 3-constrained classes, and the associated standard type homogeneous structures, one wants to know, with specific exceptions, that the pattern of forbidden triangles in any homogeneous structure of the same type defines one of the specified 3-constrained classes. In particular, in the case of metrically homogeneous graphs, we know the 3-constrained classes and a full proof of the classification conjecture one would naturally start with this problem: to show that the forbidden triangles in a metrically homogeneous graph of generic type form one of the specified admissible sets of constraints.

Problem 14 includes an inductive element under clause (b); that is, it leads to a classification problem where the structure induced on one of the 1-types over a parameter a is known, and is exceptional.

A related problem is the following.

PROBLEM 16. *Let* Γ *be a primitive binary structure,* $a \in \Gamma$, *and* p *a* 2*-type. Does it follow that* a^p *is primitive? Does it follow that the operation of algebraic closure is trivial?*

18.3. Concrete classification problems: overview

We now consider the classification problem for homogeneous 2-multi-tournaments and for homogeneous 3-multi-graphs. Our point of view is that of the preceding section: in other words, we first ask what the 3-constrained homogeneous structures are, and identify their variations (adding Henson constraints). We then ask whether the pattern of forbidden triangles is necessarily as in one of the 3-constrained cases. After that, one wants to work out in each case whether the structures with a specified collection of forbidden triangles are the standard structures associated to that particular collection.

In the case of 2-multi-tournaments and 3-multi-graphs, the 3-constrained homogeneous structures were already identified in the appendix to Cherlin [1998].

Our task in this section is to put the material of Parts I and II into this context, to see what has been accomplished and also what needs to be added at this point to complete the picture. We will devote the rest of the volume to filling in some of the missing pieces.

18.3.1. Classification problems for small languages. The conventional view of the distinction between graphs and tournaments, for a model theorist, is that they share the same language but different axioms. We prefer to view the language as specifying the structure of the set of quantifier-free k-types for some definite value of k (with $k = 2$ here). This structure includes the action of the symmetric group on the variables, as well as the restriction maps to ℓ-types for $\ell < k$. In the case of homogeneous tournaments and graphs, this brings us back to the point of view that tournaments are structures with one pair of anti-symmetric 2-types, and graphs are structures with two symmetric 2-types. We consider only irreflexive k-types (all elements distinct).

The number of irreflexive k-types, counted *up to symmetry*, will be called the *rank* of the language.

From this perspective, here are the known classification results that cover *all* homogeneous structures of a specified combinatorial type in a purely binary language (just one 1-type, and a finite set of irreflexive 2-types, symmetric or anti-symmetric).

- Rank 1, symmetric: Theory of equality.
- Rank 1, anti-symmetric: Tournaments: finitely many.
- Rank 2, symmetric: Graphs: countably many.
- Rank 2, with one symmetric type and one anti-symmetric pair of types: Directed graphs; uncountably many, with one uncountable family, and countably many others.

The complexity of the classification appears to rise quickly and it is unclear whether such explicit classifications can continue much further; but the only definitely known obstacle to this, so far, is the sheer length of the arguments required. It would be remarkable if one could handle all finite binary relational languages using the current methods and patience, but this is merely a more concrete way of phrasing Lachlan's question of effectivity.

We feel that the rank is more significant than the total number of 2-types; that is, while increasing the number of 2-types by breaking symmetry complicates matters substantially, increasing the rank complicates matters even more.

The "next" cases to be considered from this point of view would be the following.

(a) Rank 2, anti-symmetric.

(b) Rank 3, symmetric.

In the terminology we have adopted, structures of the first type are called 2-*multi*-*tournaments*, and structures of the second type are 3-*multi*-*graphs*, in other words tournaments with a coloring of the arcs by two colors, and complete graphs with a coloring of the edges by three colors.

The 3-constrained homogeneous structures of these types were given in Cherlin [1998], with the imprimitive ones and the free amalgamation classes omitted. This was based mainly on a computer search and no documentation was provided there. We will give these lists here, and we supply further details in the following chapters.

18.3.2. Primitive 3-constrained homogeneous 3-multi-graphs without free amalgamation. There is only one primitive 3-constrained homogeneous 3-multi-graph which is not associated with a free amalgamation class. After labeling the 2-types appropriately it may be interpreted as being either of the following metrically homogeneous graphs.

$$\Gamma^3_{3,3,10,11} \text{ or } \Gamma^3_{1,3,8,9}.$$

The forbidden triangles are then of the following types in the two cases.

$$\Gamma^3_{3,3,10,11}: (1,1,3), (2,2,1), (1,1,1);$$
$$\Gamma^3_{1,3,8,9}: (3,3,2), (1,1,3), (3,3,3).$$

The correspondence between these two points of view is given by cyclic permutation of the labels on the 2-types: $(1,3,2)$. Notice that the triangle inequality on one side corresponds to a less trivial constraint on the other side.

There are other primitive metrically homogeneous graphs of diameter 3, but the corresponding amalgamation classes have free amalgamation, using distance 2 as the "default" value. (Typically free amalgamation is interpreted as the absence of additional relations, but in binary languages with all 2-types treated on an equal footing, the meaning is that a particular type is to be used to make the amalgam; and it is preferable that the type used be symmetric.)

The infinite imprimitive homogeneous 3-multi-graphs were classified in Cherlin [1999], and all finite homogeneous 3-multi-graphs were classified by Lachlan [1986, §2], with more details given in an unpublished work (Lachlan [ca. 1982]).

We will give the catalog of all known homogeneous 3-multi-graphs in §19.1.

18.3.3. Primitive 3-constrained homogeneous 2-multi-tournaments without free amalgamation. There is a distinctly richer supply of primitive 3-constrained 2-multi-tournaments which do not correspond to free amalgamation classes. This is shown in Table 18.1, p. 10, which shows the (non-degenerate) possibilities up to a permutation of the language. The table has been arranged and labeled to suit our present purpose. We use the labels 1, 2 for the two colors of arc, and we list the forbidden triangles using the following conventions.

NOTATION 18.3.1. For $i, j, k \in \{1,2\}$, the symbol "$C_3(i,j,k)$" denotes an oriented 3-cycle with arcs of type (i,j,k) respectively: $a \xrightarrow{i} b \xrightarrow{j} c \xrightarrow{k} a$ (so this is also denoted $C_3(j,k,i)$ and $C_3(k,i,j)$).

Similarly, $L_3(i,j,k)$ is a transitive tournament on three vertices a,b,c where $a \xrightarrow{i} b \xrightarrow{j} c$ and $a \xrightarrow{k} c$.

When listing forbidden triangles we use a compressed notation, e.g., "C_3 : 111, 112" stands for "$C_3(1,1,1)$, $C_3(1,1,2)$."

Note in particular that when there is a definable linear order it may be taken to be given by $\xrightarrow{1} \cup \xrightarrow{2}$; there are four such cases, numbered 2–5. These are the cases in which the structure may be viewed as an ordered graph, or as an ordered tournament.

This catalog will be discussed in Chapter 22. The label "Exceptional" means "poorly understood" and, in particular, the existence of those structures is one of the points we will need to check.

#	*Constraints*		*Type*
	Group I: *Finite*		
1	C_3: 111,112,222	L_3: 111,121,122,211, 212,221,222	Pentagram
	Group II: *Linearly ordered by* $\xrightarrow{1} \cup \xrightarrow{2}$		
	Common Constraints		
	C_3: 111,112,221,222	L_3: None	
	Additional Constraints		
2	C_3: none	L_3: none	RG $*$ $<$
3	C_3: none	L_3: 112	$\leq$ extends PO
4	C_3: none	L_3: 111	Henson $*$ $<$
5	C_3: none	L_3: 112,221	$< * <$
	Group III: *p.o. by* $\xrightarrow{1}$, *no definable linear order*		
	Common Constraints		
	C_3: 111, 112	L_3: 112	
6	C_3: none	L_3: none	Desymm. PO
7	C_3: 221	L_3: none	Exceptional
	Group IV: *Infinite, no definable partial order*		
	Constraints		
8	C_3: 111,112	L_3: none	Exceptional
9	C_3: 112	L_3: 111	"
10	C_3: 111,112	L_3: 111	"
11	C_3: 111,222	L_3: 111,122,212	$\widetilde{\mathbb{S}(3)}$
12	C_3: 111,112	L_3: 121,211,221,222	$\mathbb{S}(4)$

TABLE 18.1. Primitive 3-constrained non-degenerate homogeneous 2-multi-tournaments without free amalgamation.

18.3.4. The story so far: 3-constraint. Recall the standard plan of attack for classification theorems in binary languages.

1. Determine the 3-constrained structures.
2. Determine the associated standard structures.
3. Attempt to show that the set of forbidden triangles in a homogeneous structure taken by itself already determines an amalgamation class, on the list above. Note any exceptions (aiming for a complete classification of all such at this stage).
4. Attempt to show, ideally, that every homogeneous structure in the class is standard, in the sense corresponding to the associated 3-constrained