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Introduction to Superconductivity

1.1 Basic Properties of Superconductivity

Superconductivity, as an emergent macroscopic quantum phenomenon, is one of the most important subjects of contemporary condensed matter physics. It was first discovered by Dutch physicist Heike Kamerlingh Onnes on April 8, 1911 [8–10]. In 1908, Onnes and his assistants successfully liquefied helium and for the first time reached low temperatures below 4.25K. This was a historic breakthrough for low temperature physics. When they applied this technique and measured the resistance of mercury, they found that its resistance dropped abruptly from 0.1 Ω to below 10^{-6} Ω within a narrow temperature range of 0.01 K around 4.2 K. This important discovery opened up the field of superconductivity and related applications. It also greatly stimulated the study of quantum emergent phenomena in condensed matter physics.

Understanding the phenomena and exploring the mechanism of superconductivity are historically important in the development of condensed matter physics. In the early days, condensed matter physics was not considered as fundamental as quantum field theory by the mainstream of physics. Various classical and quantum mechanical theories were developed to study solid state phenomena, such as the Drude theory of transport, the Sommerfeld theory of electrons, the Debye theory of phonons, and the Bloch theory of energy band structures. However, there were few original fundamental principles arising from this field. This situation was changed when the mechanism of superconductivity as well as that of superfluidity was revealed.

A superconductor has two characteristic electromagnetic features, namely zero direct current resistance and perfect diamagnetism. Zero resistance means that superconductors are ideal conductors, and there is no energy loss during electric energy transport using superconducting transmission lines. Moreover, superconductors are more than just ideal conductors. More fundamentally, superconductors exhibit perfect diamagnetism which expels magnetic flux lines from the interior of superconductor. The external magnetic field can only penetrate into superconductors within a short length scale near the surface called the penetration length. The perfect diamagnetism of superconductivity was discovered by W. Meissner and R. Ochsenfeld in 1933. It is also called the Meissner effect [11]. The Meissner effect is *not* a consequence of zero resistance but an independent fundamental property resulting from the phase coherence of superconductivity.

The Meissner effect distinguishes a superconductor from an ideal normal conductor in their responses to an applied magnetic field. If a magnetic field is applied to a normal metal, Faraday's law, or Lentz's law, says that a screening eddy current is induced to expel the magnetic flux. However, due to the existence of resistance, the induced eddy current dissipates and eventually decays to zero, allowing the magnetic field to penetrate into the interior of the conductor. On the other hand, if the magnetic field is applied to an ideal conductor or a superconductor at low temperatures, as there is no resistance in either case, a persistent eddy current exists which expels the magnetic field from within the bulk. Now if the temperature is raised so that both systems return back to their normal metallic states, the magnetic field penetrates to the bulks again. So far we have not seen any difference between a superconductor and an ideal conductor.

A sharp contrast between an ideal conductor and a superconductor appears when both systems are cooled down. In an ideal conductor, the magnetic field remains inside the system, while in a superconductor, the magnetic field is expelled to the outside. Thus, for an ideal conductor, it matters if it is field cooled or zero field cooled, whereas for a superconductor, regardless of the external field and its history, the magnetic field becomes zero inside the bulk.

The zero resistance and the Meissner effect are two defining properties of superconductors that cannot be understood in the framework of the single-electron theory, or, the band theory. In the macroscopic world, dissipation and friction are nearly unavoidable. How can electric currents be free of dissipation? Diamagnetism is found in nearly all materials, but it is generally very weak and can only be observed in materials that do not exhibit other forms of magnetism. The perfect diamagnetism exhibited in superconductors is even more puzzling than the appearance of zero resistance. Quite a number of noble metals, such as gold, silver, and cooper, are in fact not superconducting at all at ambient pressure. Thus superconductivity is not a consequence of weak dissipation. Instead, it is a macroscopic phenomenon, resulting from the collective interplay of electrons.

The superconducting state is a distinct thermodynamic phase. It occurs when the temperature is reduced below a critical temperature, denoted as T_c, through a second order phase transition in the absence of an external magnetic field. The superconducting transition temperatures are generally below 25 K. High temperature superconductors are ideally defined as materials that superconduct at temperatures above the boiling point of liquid nitrogen, i.e. 77 K. However, in the literature, materials with T_c close to or larger than 40 K are all referred to as high-T$_c$ superconductors. The high-T$_c$ superconductors that have been discovered include: (1) hole or electron doped perovskite copper oxides, first discovered by Bednorz and Müller in 1986 [1]; (2) electron or hole doped iron pnictides [5] or chalcogenides [12], first discovered by Hosono and coworkers in 2008; (3) superhydride compounds under ultrahigh pressure, anticipated by Ashcroft for metallic hydrogen [13] and hydrogen enriched materials [14], and first confirmed experimentally in H_3S by Drozdov et al. in 2015 [15]; (4) Magnesium diboride with $T_c \sim 39$ K, discovered by Nagamatsu et al. in 2001 [16]. The current highest T_c record holder is $HgBa_2Ca_2Cu_3O_{8+\delta}$ (133 K) at ambient pressure [17], and carbonaceous sulfur hydride (288 K) under 267 GPa [18].

The phase transition from a normal metallic or insulating state to a superconducting state corresponds to the formation of superconducting long-range order. Different from ferromagnetism, the superconducting order is an off-diagonal long-range order which does not have a classical correspondence [19]. By lowering temperatures, there exists a critical temperature range within which the resistance drops to zero. The width of this critical region is determined by the fluctuation of superconducting order parameter. In conventional metal-based superconductors, this critical temperature range is very narrow, and the resistance drops to zero abruptly. However, in high-T$_c$ copper oxides or iron-based superconductors, or in dirty superconductors of metals and alloys, fluctuations are strong. The corresponding critical regions are broad and the resistance drops are relatively slow.

A superconductor has exactly zero direct-current resistance and is able to maintain an electric current without generating an external voltage in the superconducting state. It loses the superconducting phase coherence and exhibits a small but finite resistance in the presence of an alternative current. One can also turn a superconductor into a normal conductor by applying a strong magnetic field or a direct electric current. For a given temperature, the highest applied magnetic field or electric current under which a material remains superconducting are called the upper critical field or the critical current.

1.2 Two-Fluid Model and London Equations

Historically, an important phenomenological theory of superconductivity is the two-fluid model first proposed by Groter and Casimir [20]. The key assumption of this model is the existence of two different types of electrons in superconductors, namely normal and superconducting electrons. The density of normal electrons is called the normal fluid density and that of superconducting electrons is called the superfluid density. The sum of these two kinds of densities gives the total density of electrons. Normal electrons carry entropy and behave similarly as in ordinary metals. Their states are changed by scattering with phonons and impurities. In contrast, superconducting electrons are resistance free. They do not carry entropy and have no contribution to thermodynamic quantities such as the specific heat. A static electric field cannot exist in an equilibrium superconducting state. Otherwise, superconducting electrons would be accelerated without attenuation, leading to a divergent electric current. The existence of superconducting electrons with zero electric field explains why the resistance is zero. However, the two-fluid model does not answer the question of how superconducting electrons are formed, neither can it explain the Meissner effect.

In order to explain the Meissner effect, Fritz and Hentz London brothers proposed an electromagnetic equation [21] to describe the superconducting current. This equation connects the superconducting current density $\mathbf{J}_s$ with the electromagnetic vector potential $\mathbf{A}$. Under the Coulomb gauge (also known as the transverse gauge) where $\nabla \cdot \mathbf{A} = 0$, it can be expressed as

$$\mathbf{J}_s = -\frac{n_s e^2}{m}\mathbf{A}, \tag{1.1}$$

where n_s is the superfluid density of electrons. This equation is called the London equation. It cannot be deduced from the Maxwell equations and should be viewed as an independent electromagnetic equation by treating superconductors as a special class of electromagnetic media.

The London equation could be rigorously derived only after the microscopic theory of superconductivity has been established. For better understanding its physical meaning, a heuristic argument is commonly given to formally "derive" this equation within theory of classical electromagnetism. A basic assumption is that electrons are moving in a frictionless state, so that

$$m\dot{\mathbf{v}}_s = -e\mathbf{E}, \tag{1.2}$$

where $\mathbf{v}_s$ is the velocity of superconducting electrons and $\mathbf{E}$ is the electric field. The supercurrent $\mathbf{J}_s = -en_s\mathbf{v}_s$ is then governed by the equation

$$\frac{\partial \mathbf{J}_s}{\partial t} = \frac{e^2 n_s}{m}\mathbf{E}, \tag{1.3}$$

which is referred to as the first London equation. Then, using the Maxwell equation,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.4}$$

we immediately arrive at

$$\frac{\partial}{\partial t}\left(\nabla \times \mathbf{J}_s + \frac{e^2 n_s}{m}\mathbf{B}\right) = 0. \tag{1.5}$$

This describes the behavior of an ideal conductor. To describe the Meissner effect, the constant of integration must be chosen to zero so that

$$\nabla \times \mathbf{J}_s + \frac{e^2 n_s}{m}\mathbf{B} = 0. \tag{1.6}$$

This is the second London equation.

The two London equations can be combined into a single one, i.e. Eq. (1.1), in terms of the vector potential in the Coulomb gauge. One can also write the London equation in an arbitrarily chosen gauge. In that case, the London equation becomes

$$\mathbf{J}_s = \frac{e^2 n_s}{m}\left(-\mathbf{A} + \nabla\varphi\right), \tag{1.7}$$

which differs from Eq. (1.1) by a gradient of a scalar field φ. Later on, we will see that this scalar field is just the condensation phase field and the corresponding term reflects the nonlocal effect of electromagnetic responses. $\nabla\varphi$ is to shift the vector potential from an arbitrary gauge to the Coulomb gauge.

If the second London equation is manipulated by applying Ampere's law,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_s, \tag{1.8}$$

it turns into the Helmholtz equation for the magnetic field:

$$\nabla^2 \mathbf{B} = \frac{\mu_0 n_s e^2}{m}\mathbf{B}. \tag{1.9}$$

In a semi-infinite plate of superconductor with its surface perpendicular to the x-direction, the solution of Eq. (1.9) is simply given by

$$B(x) = B(x_0)e^{-(x-x_0)/\lambda}, \tag{1.10}$$

where

$$\lambda = \sqrt{\frac{m}{\mu_0 n_s e^2}} \tag{1.11}$$

is the London penetration depth describing the decay length of an external magnetic field and x_0 is the x-coordinate of the superconductor-vacuum interface. In the limit $x - x_0 \gg \lambda$, the magnetic field decays to zero. This gives a phenomenological explanation to the Meissner effect.

In spite of its simplicity, the two-fluid model captures the key features of superconductors. The key concepts – the normal and superconducting electrons – were broadly used in the construction of the microscopic theory of superconductivity. The normal and superconducting electrons correspond to the quasiparticle excitations and the superconducting paired electrons, respectively. The two-fluid model has played an important role in the study of superconductivity, although it does not explain the microscopic mechanism of superconductivity. Even after the establishment of the microscopic theory of superconductivity, it is still useful to apply the two-fluid model to understand qualitatively experimental results of superconductors.

1.3 Cooper Pairing

Superconductivity is a quantum many-body effect and cannot be understood based on the single-electron theory and its perturbative expansion. In 1956, Cooper considered a two-electron problem which turned out to be one of the most crucial steps toward a microscopic understanding of superconductivity [22]. He showed that if there exists an effective attraction interaction, no matter how weak it is, between two electrons in a background of the Fermi sea, the Fermi surface is no longer stable. Electrons on the Fermi surface will pair each other to form bound states, so that the ground state energy is reduced. The bound state of paired electrons is called a Cooper pair.

The Cooper instability results from the interplay between the weak attractive interaction and the Fermi sea. The appearance of the Fermi sea is crucial. Otherwise, the Cooper pairing instability would not happen in an arbitrarily weak attractive potential. In free space, two electrons can form a bound state only if the attractive interaction between them is sufficiently strong (above a finite threshold) in three dimensions.

The proof given by Cooper is based on a simple variational calculation. He considered how the ground state energy is changed by adding two extra electrons with opposite momenta and spins to a filled Fermi sea at zero temperature. Due to the Pauli exclusion principle, these two electrons can only be put outside the Fermi sea. For simplicity in the calculation, he assumed that the attractive potential is nonzero only when both electrons lie between the Fermi energy E_F and $E_F + \omega_D$, and the amplitude of the potential V_0 is momentum independent. Here the cutoff ω_D is a characteristic energy scale determined by the mechanism or resource from which the attraction is induced. If the effective attraction is induced by the electron–phonon interaction, ω_D is just the characteristic frequency of phonons, namely the Debye frequency. After a simple variational calculation, Cooper found that the two electrons form a bound state with the binding energy

$$\Delta E = 2\Delta = -2\hbar\omega_D e^{-2/N_F g}, \tag{1.12}$$

where N_F is the electron density of states on the Fermi surface, and g is the coupling strength. This is also the energy needed to break a Cooper pair. This result shows that the Fermi surface is unstable against a small attractive interaction. It also reveals two important parameters in describing a superconducting state. One is the characteristic attraction energy scale ω_D, and the other is the dimensionless coupling constant defined by the product of the density of states at the Fermi level and the depth of the attractive interaction. As discussed later, these two parameters also determine the superconducting transition temperature T_c. The calculation made by Cooper is simple, but it captured the main character of superconductivity.

Equation (1.12) shows that the dependence of the binding energy on the interaction strength g is singular. It implies that the microscopic theory of superconductivity cannot be established through perturbative calculations based on normal conducting states. This is actually the major difficulty in the study of the superconducting mechanism, which obstructed the development of a microscopic theory of superconductivity for nearly fifty years after its discovery.

To see more clearly how the Cooper pairing energy comes about, let us follow Cooper to solve a simple model of two electrons added to a rigid Fermi sea at zero temperature. It is assumed that the two electrons interact with each other but not with those in the Fermi sea. To reduce the repulsive interaction applied by the exclusion principle, the two electrons should form a spin singlet so that their spin wave function is antisymmetric and their spatial wave function is symmetric. Moreover, the lowest energy state should have zero total momentum so that the electrons must have opposite momenta. Therefore, the wave function has the form

$$|\Psi\rangle = \sum_{\mathbf{k}} \alpha(\mathbf{k}) c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k}\downarrow} |0\rangle, \tag{1.13}$$

where $|0\rangle$ is the vacuum composed of the rigid Fermi sea.

This interacting system of two electrons is governed by the Hamiltonian

$$H = \sum_{\mathbf{k}} (\varepsilon_{\mathbf{k}} - \mu) c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k},\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} c^{\dagger}_{\mathbf{k}\uparrow} c^{\dagger}_{-\mathbf{k},\downarrow} c_{-\mathbf{k}',\downarrow} c_{\mathbf{k}',\uparrow}, \tag{1.14}$$

where $\varepsilon_{\mathbf{k}}$ is the energy dispersion of electrons and μ is the chemical potential. $V_{\mathbf{k},\mathbf{k}'}$ is the scattering potential between two Cooper pairs with momenta $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ and $(\mathbf{k}'\uparrow, -\mathbf{k}'\downarrow)$. For simplicity, the attractive interaction between the two electrons is assumed to be momentum independent and to take a simple form

$$V_{\mathbf{k},\mathbf{k}'} = \frac{g}{V}, \tag{1.15}$$

with V the system volume. From the Schrödinger equation

$$H|\Psi\rangle = E|\Psi\rangle, \tag{1.16}$$

we find the equation that $\alpha(\mathbf{k})$ satisfies

$$2\xi_{\mathbf{k}}\alpha(\mathbf{k}) - \frac{g}{V}\sum_{\mathbf{k}'}\alpha(\mathbf{k}') = (E - E_0)\alpha(\mathbf{k}), \tag{1.17}$$

where E_0 is the energy of the filled Fermi sea and

$$\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu. \tag{1.18}$$

Equation (1.17) can be rewritten as

$$\alpha(\mathbf{k}) = \frac{g}{2\xi_{\mathbf{k}} - \Delta E} \frac{1}{V} \sum_{\mathbf{k}'} \alpha(\mathbf{k}'), \tag{1.19}$$

where $\Delta E = E - E_0$ is the energy gap of the system with respect to the vacuum. Summing over all momentum points allows $\alpha(\mathbf{k})$ to be cancelled out from both sides. This leads to the gap equation

$$\frac{1}{g} = \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{2\xi_{\mathbf{k}} - \Delta E} = N_F \int_0^{\hbar\omega_D} d\xi \frac{1}{2\xi - \Delta E}. \tag{1.20}$$

By solving this equation, we find that

$$\frac{1}{g} = \frac{N_F}{2} \ln \frac{2\hbar\omega_D - \Delta E}{-\Delta E} \approx \frac{N_F}{2} \ln \frac{2\hbar\omega_D}{|\Delta E|}. \tag{1.21}$$

This yields the result shown in Eq. (1.12).

1.4 BCS Mean Field Theory

In 1957, John Bardeen, Leon Cooper, and John Robert Schrieffer (BCS) proposed the microscopic theory of superconductivity based on the concept of Cooper pairing [23]. Their work established a fundamental theory of superconductivity. It also provided tremendous progress toward the understanding of microscopic quantum world.

In the BCS framework, there are two preconditions for the formation of superconducting condensation. The first is the formation of Cooper pairs through an attraction interaction. The second is the development of phase coherence among Cooper pairs. Cooper pairing refers to the process that electrons near the Fermi surface form bound states. It is a prerequisite of superconductivity because Cooper pairs carry the feature of bosons that eliminates the effective repulsion induced by the Fermi statistics of electrons, and can condense into a superfluid state by forming phase coherence. Cooper pairs are found to exist in all superconductors discovered so far. This gives strong support to the BCS theory.

The BCS work is a variational theory. It is based on the BCS variational wavefunction first proposed by Schrieffer. This wavefunction generalizes the solution of Cooper pair to a many-body system. It captures the main picture of Cooper for the superconducting condensation of paired electrons. The BCS theory is equivalent to the mean-field theory later developed based on the Bogoliubov transformation.

This mean-field theory is to take the Gaussian or saddle-point approximation in the framework of quantum field theory. It handles the thermal average of operators, rather than the variational wavefunction of the ground state. Fluctuations of Cooper pairs around the saddle point can be included, for example, by taking the one-loop expansion in the path-integral formulism.

The BCS mean field theory starts by considering the reduced pairing Hamiltonian defined by Eq. (1.14). This Hamiltonian is a simplification to the complex interactions of electrons. It highlights the interaction in the pairing channel and neglects interactions in other channels.

Equation (1.14) is applicable to superconductors with spin singlet pairing. It can be extended to describe spin triplet superconductors with slight modifications. This Hamiltonian considers the Cooper pairs with zero center-of-mass momentum, and neglects the pairing with finite center-of-mass momentum. The zero momentum pairing is physically reasonable because the phase space for the finite momentum pairing is strongly constrained by the Fermi surface geometry and by the momentum conservation [6]. In an external magnetic field, the Fermi surfaces of up- and down-spin electrons are split, and the pairing with finite center-of-mass momentum is favored. Cooper pairs in a current-carrying superconducting state have finite pairing momenta. But the pairing energy is suppressed and becomes zero when the current exceeds a critical current.

To define

$$A = \sum_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}, \tag{1.22}$$

we can rewrite the BCS reduced Hamiltonian as

$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \frac{g}{V} A^\dagger A. \tag{1.23}$$

Taking the mean-field approximation for the interaction term,

$$-A^\dagger A = -\langle A^\dagger \rangle A - \langle A \rangle A^\dagger + \langle A^\dagger \rangle \langle A \rangle, \tag{1.24}$$

we obtain the BCS mean-field Hamiltonian

$$H_{MF} = \sum_{\mathbf{k}} \left(\sum_{\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \Delta c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} + \Delta^* c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \right) + \frac{V}{g} |\Delta|^2. \tag{1.25}$$

$\langle A \rangle$ represents the expectation value of operator A. Δ is the superconducting order parameter determined by the equation

$$\Delta = -\frac{g}{V} \langle A \rangle = -\frac{g}{V} \sum_{\mathbf{k}} \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle. \tag{1.26}$$

$\langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle$ depends on the value of Δ.

Equation (1.26) is just the celebrated BCS gap equation. It determines completely the low energy quasiparticle excitation spectra in the superconducting state. By solving this equation self-consistently, one can determine all the thermodynamic quantities.

H_{MF} does not conserve the particle number, but the total spin, $\sum_{\mathbf{k}} \sigma c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma}$, and the total momentum of the Cooper pairs remain conserved. H_{MF} can be diagonalized by a unitary matrix using the Bogoliubov transformation introduced in Appendix A

$$\begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} = \begin{pmatrix} u_{\mathbf{k}} & v_{\mathbf{k}} \\ -v^*_{\mathbf{k}} & u^*_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} \alpha_{\mathbf{k}} \\ \beta^\dagger_{\mathbf{k}} \end{pmatrix}. \tag{1.27}$$

After the diagonalization, the Hamiltonian becomes

$$H_{MF} = \sum_{\mathbf{k}} E_{\mathbf{k}} \left(\alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}} + \beta^\dagger_{\mathbf{k}} \beta_{\mathbf{k}} \right) + \sum_{\mathbf{k}} (\xi_{\mathbf{k}} - E_{\mathbf{k}}) + \frac{V}{g} \Delta^2. \tag{1.28}$$

$\alpha^\dagger_{\mathbf{k}}$ and $\beta^\dagger_{\mathbf{k}}$ are the creation operators of the Bogoliubov quasiparticles. They describe the single-particle excitations above the superconducting gap, corresponding to the normal electrons in the two-fluid model. The quasiparticle excitation energy is given by

$$E_{\mathbf{k}} = \sqrt{\xi^2_{\mathbf{k}} + \Delta^2}. \tag{1.29}$$

On the Fermi surface, $\xi_{\mathbf{k}} = 0$ and $E_{\mathbf{k}} = |\Delta|$. Thus $\Delta_{\mathbf{k}}$ is the gap function of quasiparticles in momentum space. The matrix elements $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ satisfy the normalization condition, $u^2_{\mathbf{k}} + v^2_{\mathbf{k}} = 1$, and are determined by

$$u_{\mathbf{k}} = \sqrt{\frac{1}{2} + \frac{\xi_{\mathbf{k}}}{2E_{\mathbf{k}}}}, \tag{1.30}$$

$$v_{\mathbf{k}} = -\mathrm{sgn}(\Delta) \sqrt{\frac{1}{2} - \frac{\xi_{\mathbf{k}}}{2E_{\mathbf{k}}}}. \tag{1.31}$$

By calculating the pairing correlation function using the above solution, we can express explicitly the gap equation as

$$1 = \frac{g}{V} \sum_{\mathbf{k}} \frac{1}{2E_{\mathbf{k}}} \tanh \frac{\beta E_{\mathbf{k}}}{2}. \tag{1.32}$$

The temperature dependence of the energy gap can be determined by self-consistently solving this equation. Moreover, the superconducting transition temperature T_c can be solved from this equation by setting $\Delta = 0$.

At zero temperature, there are no quasiparticle excitations, and both $\langle \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}} \rangle$ and $\langle \beta^\dagger_{\mathbf{k}} \beta_{\mathbf{k}} \rangle$ are zero. The ground state wavefunction can be obtained by projecting out