CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS 200

Editorial Board
J. BERTOIN, B. BOLLOBÁS, W. FULTON, B. KRA, I. MOERDIJK, C. PRAEGER, P. SARNAK, B. SIMON, B. TOTARO

SCHRÖDINGER OPERATORS: EIGENVALUES AND LIEB–THIRRING INEQUALITIES

The analysis of eigenvalues of Laplace and Schrödinger operators is an important and classical topic in mathematical physics with many applications. This book presents a thorough introduction to the area, suitable for masters and graduate students, and includes an ample amount of background material on the spectral theory of linear operators in Hilbert spaces and on Sobolev space theory.

Of particular interest is a family of inequalities by Lieb and Thirring on eigenvalues of Schrödinger operators, which they used in their proof of stability of matter. The final part of this book is devoted to the active research on sharp constants in these inequalities and contains state-of-the-art results, serving as a reference for experts and as a starting point for further research.

Rupert L. Frank holds a chair in applied mathematics at LMU Munich and is doing research primarily in analysis and mathematical physics. He is an invited speaker at the 2022 International Congress of Mathematics.

Ari Laptev is Professor at Imperial College London. His research interests include different aspects of spectral theory and functional inequalities. He is a member of the Royal Swedish Academy of Science, a Fellow of EurASc and a member of Academia Europaea.

Timo Weidl is Professor at the University of Stuttgart. He works on spectral theory and mathematical physics.
CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

Editorial Board
J. Bertoin, B. Bollobás, W. Fulton, B. Kra, I. Moerdijk, C. Praeger, P. Sarnak, B. Simon, B. Totaro

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing, visit www.cambridge.org/mathematics.

Already Published

162 C. J. Bishop & Y. Peres Fractals in Probability and Analysis
163 A. Bovier Gaussian Processes on Trees
164 P. Schneider Galois Representations and \( (\varphi, \Gamma) \)-Modules
166 D. Li & H. Queffélec Introduction to Banach Spaces, I
167 D. Li & H. Queffélec Introduction to Banach Spaces, II
169 J. M. Landsberg Geometry and Complexity Theory
170 J. S. Milne Algebraic Groups
171 J. Gough & J. Kupsch Quantum Fields and Processes
172 T. Ceccherini-Silberstein, F. Scarabotti & F. Tolli Discrete Harmonic Analysis
173 P. Garrett Modern Analysis of Automorphic Forms by Example, I
174 P. Garrett Modern Analysis of Automorphic Forms by Example, II
175 G. Navarro Character Theory and the McKay Conjecture
176 P. Feig, H. P. A. Gustafsson, A. Kleinschmidt & D. Persson Eisenstein Series and Automorphic Representations
177 E. Peterson Formal Geometry and Bordism Operators
178 A. Ogus Lectures on Logarithmic Algebraic Geometry
179 N. Nikolov Hardy Spaces
180 D.-C. Cisinski Higher Categories and Homotopical Algebra
181 A. A-grachev, D. Barili & U. Bosco A Comprehensive Introduction to Sub-Riemannian Geometry
182 N. Nikolov Toeplitz Matrices and Operators
183 A. Yekutieli Derived Categories
184 C. Demeter Fourier Restriction, Decoupling and Applications
185 D. Barnes & C. Roitzen Foundations of Stable Homotopy Theory
186 V. Vasyunin & A. Volberg The Bellman Function Technique in Harmonic Analysis
187 M. Geck & G. Malle The Character Theory of Finite Groups of Lie Type
188 B. Richter Category Theory for Homotopy Theory
189 R. Willet & G. Yu Higher Index Theory
190 A. Bobrowski Generators of Markov Chains
191 D. Zhao, S. Peng & S. Yan Singularly Perturbed Methods for Nonlinear Elliptic Problems
192 E. Kowalski An Introduction to Probabilistic Number Theory
193 Y. Gorin Lectures on Random Lozenge Tilings
194 E. Richl & D. Verity Elements of \( \infty \)-Category Theory
195 H. Krause Homological Theory of Representations
196 F. Durand & D. Perrin Dimension Groups and Dynamical Systems
197 A. Shiferaw Polynomial Methods and Incidence Theory
198 T. Dobson, A. Malnič & D. Marušič Symmetry in Graphs
199 K. S. Kedlaya p-adic Differential Equations
201 J. van Neerven Functional Analysis
Schrödinger Operators: Eigenvalues and Lieb–Thirring Inequalities

RUPERT L. FRANK
Ludwig-Maximilians-Universität München

ARI LAPTEV
Imperial College of Science, Technology and Medicine, London

TIMO WEIDL
Universität Stuttgart
To
Semra, Sami and Sima

To
Marilyn, Maria, Vanya, Katya and Eugenia

To
Galia, Adi and Alex
Contents

Preface  page xi
Overview  1

PART ONE  BACKGROUND MATERIAL  5

1 Elements of Operator Theory  7
  1.1 Hilbert spaces, self-adjoint operators and the spectral theorem  8
  1.2 Semibounded operators and forms, and the variational principle  29
  1.3 Comments  64

2 Elements of Sobolev Space Theory  66
  2.1 Weak derivatives  67
  2.2 Sobolev spaces  85
  2.3 Compact embeddings  96
  2.4 Sobolev inequalities on the whole space  103
  2.5 Friedrichs and Poincaré inequalities  112
  2.6 Hardy inequalities  127
  2.7 Homogeneous Sobolev spaces  142
  2.8 The extension property  146
  2.9 Comments  157

PART TWO  THE LAPLACE AND SCHRÖDINGER OPERATORS  165

3 The Laplacian on a Domain  167
  3.1 The Dirichlet and Neumann Laplacians  170
  3.2 Weyl’s asymptotic formula for the Dirichlet Laplacian  179
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.3</td>
<td>Weyl’s asymptotic formula for the Neumann Laplacian</td>
<td>192</td>
</tr>
<tr>
<td>3.4</td>
<td>Pólya’s inequality for tiling domains</td>
<td>197</td>
</tr>
<tr>
<td>3.5</td>
<td>Lower bounds for the eigenvalues of the Dirichlet Laplacian</td>
<td>199</td>
</tr>
<tr>
<td>3.6</td>
<td>Upper bounds for the eigenvalues of the Neumann Laplacian</td>
<td>209</td>
</tr>
<tr>
<td>3.7</td>
<td>Phase space interpretation</td>
<td>212</td>
</tr>
<tr>
<td>3.8</td>
<td>Appendix: The Laplacian in spherical coordinates</td>
<td>219</td>
</tr>
<tr>
<td>3.9</td>
<td>Comments</td>
<td>232</td>
</tr>
<tr>
<td>4</td>
<td>The Schrödinger Operator</td>
<td>244</td>
</tr>
<tr>
<td>4.1</td>
<td>Definition of the Schrödinger operator</td>
<td>247</td>
</tr>
<tr>
<td>4.2</td>
<td>Explicitly solvable examples</td>
<td>251</td>
</tr>
<tr>
<td>4.3</td>
<td>Basic spectral properties of Schrödinger operators</td>
<td>265</td>
</tr>
<tr>
<td>4.4</td>
<td>Weyl’s asymptotic formula for Schrödinger operators</td>
<td>279</td>
</tr>
<tr>
<td>4.5</td>
<td>The Cwikel–Lieb–Rozenblum inequality</td>
<td>282</td>
</tr>
<tr>
<td>4.6</td>
<td>The Lieb–Thirring inequality</td>
<td>292</td>
</tr>
<tr>
<td>4.7</td>
<td>Extending inequalities and asymptotics</td>
<td>298</td>
</tr>
<tr>
<td>4.8</td>
<td>Reversed Lieb–Thirring inequality</td>
<td>303</td>
</tr>
<tr>
<td>4.9</td>
<td>Comments</td>
<td>308</td>
</tr>
<tr>
<td></td>
<td><strong>PART THREE</strong> SHARP CONSTANTS IN LIEB–THIRRING INEQUALITIES</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Sharp Lieb–Thirring Inequalities</td>
<td>319</td>
</tr>
<tr>
<td>5.1</td>
<td>Basic facts about Lieb–Thirring constants</td>
<td>321</td>
</tr>
<tr>
<td>5.2</td>
<td>Lieb–Thirring inequalities for special classes of potentials</td>
<td>323</td>
</tr>
<tr>
<td>5.3</td>
<td>The sharp bound for $\gamma = \frac{1}{2}$ in one dimension</td>
<td>335</td>
</tr>
<tr>
<td>5.4</td>
<td>The sharp bound for $\gamma = \frac{3}{2}$ in one dimension</td>
<td>349</td>
</tr>
<tr>
<td>5.5</td>
<td>Trace formulas for one-dimensional Schrödinger operators</td>
<td>353</td>
</tr>
<tr>
<td>5.6</td>
<td>Comments</td>
<td>362</td>
</tr>
<tr>
<td>6</td>
<td>Sharp Lieb–Thirring Inequalities in Higher Dimensions</td>
<td>385</td>
</tr>
<tr>
<td>6.1</td>
<td>Schrödinger operators with matrix-valued potentials</td>
<td>388</td>
</tr>
<tr>
<td>6.2</td>
<td>The Lieb–Thirring inequality with the semiclassical constant</td>
<td>395</td>
</tr>
<tr>
<td>6.3</td>
<td>The sharp bound in the matrix-valued case for $\gamma = \frac{3}{2}$</td>
<td>400</td>
</tr>
<tr>
<td>6.4</td>
<td>Trace formulas in the matrix case</td>
<td>411</td>
</tr>
<tr>
<td>6.5</td>
<td>Comments</td>
<td>414</td>
</tr>
<tr>
<td>7</td>
<td>More on Sharp Lieb–Thirring Inequalities</td>
<td>417</td>
</tr>
<tr>
<td>7.1</td>
<td>Monotonicity with respect to the semiclassical parameter</td>
<td>418</td>
</tr>
<tr>
<td>7.2</td>
<td>Bounds for radial potentials</td>
<td>426</td>
</tr>
</tbody>
</table>
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.3 More on the one-particle constants</td>
<td>434</td>
</tr>
<tr>
<td>7.4 The dual Lieb–Thirring inequality</td>
<td>436</td>
</tr>
<tr>
<td>7.5 Comments</td>
<td>448</td>
</tr>
<tr>
<td>8 More on the Lieb–Thirring Constants</td>
<td>451</td>
</tr>
<tr>
<td>8.1 More on Lieb–Thirring inequalities in the matrix-valued case</td>
<td>452</td>
</tr>
<tr>
<td>8.2 The sum of the square roots of the eigenvalues</td>
<td>456</td>
</tr>
<tr>
<td>8.3 The sum of the eigenvalues</td>
<td>457</td>
</tr>
<tr>
<td>8.4 Summary on constants in Lieb–Thirring inequalities</td>
<td>463</td>
</tr>
<tr>
<td>8.5 Comments</td>
<td>468</td>
</tr>
<tr>
<td>References</td>
<td>471</td>
</tr>
<tr>
<td>Index</td>
<td>504</td>
</tr>
</tbody>
</table>
Preface

Ever since E. H. Lieb and W. Thirring published their celebrated work *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities* (Lieb and Thirring, 1976), the branch of spectral theory related to such inequalities has flourished and these bounds are now named for them.

In their 1976 paper Lieb and Thirring developed a family of inequalities, of which they had used a special case in an earlier (1975) paper, to prove stability of matter. Their approach simplified and improved the work by Lenard and Dyson (1968), and introduced fundamental ideas in the analysis of fermionic quantum many-body systems.

Lieb–Thirring inequalities come in two different versions, namely a spectral form and a kinetic (or dual) form. The kinetic form is most directly applicable to quantum many-body systems and provides a lower bound on the total kinetic energy in terms of certain simple effective characteristic of the state. It is a mathematical expression of the uncertainty and Pauli exclusion principles and a far-reaching generalization of Sobolev inequalities. The spectral form of the Lieb–Thirring inequalities concerns the negative eigenvalues $-E_j$ of the one-particle Schrödinger operator

$$H = -\Delta - V$$

and provides upper bounds on the Riesz means

$$\sum_j E_j^\gamma, \quad \gamma > 0,$$

that only involve an $L^p$ norm of $V$. In the special case $\gamma = 1$, these bounds are equivalent to the kinetic form of the Lieb–Thirring inequalities.

It turned out that this form is also very useful in the study of the dimension of attractors for the Navier–Stokes and other non-linear evolution equations.
xii

Preface

Originally, it was calculated by exploiting the available Sobolev inequalities and then employing the relationship between that dimension and the prevailing Lyapunov exponents, as first conjectured by Farmer et al. (1983). This effective, but somewhat cumbersome approach was significantly simplified by the use of the Lieb–Thirring improvement of the Sobolev inequalities (Lieb, 1984; Constantin et al., 1985; Temam, 1997).

Lieb–Thirring inequalities are also important in the study of properties of the essential spectrum of Schrödinger operators. Deift and Killip (1999) were able to obtain a sharp result on the absolute continuity of the positive spectrum for one-dimensional Schrödinger operators using a trace formula by Zaharov and Faddeev (1971); see also Killip and Simon (2009). Such trace formulas provide identities between characteristics of the spectrum (eigenvalues and scattering data) and some functionals involving electric potentials of Schrödinger operators. In addition, they describe integrals of motions of the Korteweg–De Vries (KdV) equation. It is worth noting that one such trace formula was used by Gardner et al. (1974) to prove a special case of what came to be called the Lieb–Thirring inequalities. Later a version of this trace formula for matrix-valued potentials played an important role in obtaining sharp constants in Lieb–Thirring inequalities for multi-dimensional Schrödinger operators (Laptev and Weidl, 2000b). It is remarkable that some sharp constants in Lieb–Thirring inequalities are related to soliton-type potentials appearing in the theory of the KdV equation.

Spectral inequalities in the special case $\gamma = 0$, which correspond to bounds on the number of negative eigenvalues of Schrödinger operators, have an even longer history. After initial results by Bargmann, Birman, Schwinger, Calogero and others, a systematic investigation was started in the Russian school of M. Birman and M. Solomyak and in the US by B. Simon and E. Lieb. The respective estimates are known as Cwikel–Lieb–Rozenblum (CLR) inequalities (Cwikel, 1977; Lieb, 1976, 1980; Rozenbljum, 1972a, 1976). One of the main motivations for proving such inequalities was to obtain necessary and sufficient conditions on the potentials for admitting Weyl asymptotics. By now, there are at least seven proofs of the CLR bound using rather different tools from mathematical analysis.

It has been more than four decades since the paper of Lieb and Thirring was published, and the theory of Lieb–Thirring inequalities is still a very dynamically developing area of functional analysis and mathematical physics. The main conjecture on the sharp Lieb–Thirring constant for $\gamma = 1$ in dimension three remains open. This area has become a well-established part of analysis that generates beautiful new ideas and that finds applications beyond the spectral theory of Schrödinger operators.
Preface

This book is aimed at presenting the current state of the art of some parts of spectral theory of partial differential equations. Our intention was to write a book that covers some new results connected to spectral properties of Schrödinger and Laplace operators that have been obtained during the last few decades. Most of them are focused around our own interests related to Lieb–Thirring inequalities. While writing the text and when giving courses based on preliminary versions of the book, we faced the problem that we needed a lot of material from the general spectral theory of self-adjoint operators in Hilbert spaces and a number of standard functional inequalities. Including all this has substantially increased the size of the book and forced us to postpone some core material for the future. We now hope that the book might be useful not only to our colleagues who are interested in spectral inequalities, but also to students specializing in analysis.

Acknowledgements

We would like to express our appreciation to M. Birman, E. Lieb, B. Simon and M. Solomyak who profoundly influenced our mathematical taste. We have been fortunate to have had the opportunity to collaborate with so many individuals on the topic of Lieb–Thirring and spectral inequalities. We would like to thank our teachers and students, colleagues and friends for all the inspiring discussions.

We owe a great debt to J. Peteranderl who read a preliminary version of this book line by line and found countless minor and major inaccuracies. For valuable remarks, suggestions and corrections we are very grateful to C. Dietze, F. Gesztesy, A. Ilyin, V. Kußmaul, S. Larson, M. Lewin, E. Lieb, G. Rozenblum, L. Schimmer and H. Siedentop. Our special thanks go to E. Lieb and B. Simon for their interest in this book and their constant encouragement.

We are grateful to David Tranah and the team at CUP for their assistance when publishing the book.

Parts of this book were written at Caltech, Imperial College London, KTH Stockholm, LMU Munich, Mittag–Leffler Institute, Oberwolfach Research Institute, Stuttgart University, and Tsinghua University, and we would like to thank all these institutions for their hospitality.

Partial support through US National Science Foundation grants PHY-1068285, PHY-1347399, DMS-1363432 and DMS-1954995 and through the Deutsche Forschungsgemeinschaft EXC-2111-390814868 (R.L.F.) is acknowledged.

Munich, London, Stuttgart,
August 2022

Rupert L. Frank
Ari Laptev
Timo Weidl