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SCHRÖDINGER OPERATORS: EIGENVALUES AND LIEB–THIRRING INEQUALITIES

The analysis of eigenvalues of Laplace and Schrödinger operators is an important and classical topic in mathematical physics with many applications. This book presents a thorough introduction to the area, suitable for masters and graduate students, and includes an ample amount of background material on the spectral theory of linear operators in Hilbert spaces and on Sobolev space theory.

Of particular interest is a family of inequalities by Lieb and Thirring on eigenvalues of Schrödinger operators, which they used in their proof of stability of matter. The final part of this book is devoted to the active research on sharp constants in these inequalities and contains state-of-the-art results, serving as a reference for experts and as a starting point for further research.

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Schrödinger Operators: Eigenvalues and Lieb–Thirring Inequalities

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CAMBRIDGE
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University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
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103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

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www.cambridge.org
Information on this title: www.cambridge.org/9781009218467
DOI: 10.1017/9781009218436

First published 2023

A catalogue record for this publication is available from the British Library.

ISBN 978-1-009-21846-7 Hardback

To
Semra, Sami and Sima

To
Marilyn, Maria, Vanya, Katya and Eugenia

To
Galia, Adi and Alex

Contents

Preface

Ever since E. H. Lieb and W. Thirring published their celebrated work *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities* (Lieb and Thirring, 1976), the branch of spectral theory related to such inequalities has flourished and these bounds are now named for them.

In their 1976 paper Lieb and Thirring developed a family of inequalities, of which they had used a special case in an earlier (1975) paper, to prove stability of matter. Their approach simplified and improved the work by Lenard and Dyson (1968), and introduced fundamental ideas in the analysis of fermionic quantum many-body systems.

Lieb–Thirring inequalities come in two different versions, namely a spectral form and a kinetic (or dual) form. The kinetic form is most directly applicable to quantum many-body systems and provides a lower bound on the total kinetic energy in terms of certain simple effective characteristic of the state. It is a mathematical expression of the uncertainty and Pauli exclusion principles and a far-reaching generalization of Sobolev inequalities. The spectral form of the Lieb–Thirring inequalities concerns the negative eigenvalues $-E_j$ of the one-particle Schrödinger operator

$$H = -\Delta - V$$

and provides upper bounds on the Riesz means

$$\sum_j E_j^\gamma , \qquad \gamma > 0 ,$$

that only involve an L^p norm of V. In the special case $\gamma = 1$, these bounds are equivalent to the kinetic form of the Lieb–Thirring inequalities.

It turned out that this form is also very useful in the study of the dimension of attractors for the Navier–Stokes and other non-linear evolution equations.

Originally, it was calculated by exploiting the available Sobolev inequalities and then employing the relationship between that dimension and the prevailing Lyapunov exponents, as first conjectured by Farmer et al. (1983). This effective, but somewhat cumbersome approach was significantly simplified by the use of the Lieb–Thirring improvement of the Sobolev inequalities (Lieb, 1984; Constantin et al., 1985; Temam, 1997).

Lieb–Thirring inequalities are also important in the study of properties of the essential spectrum of Schrödinger operators. Deift and Killip (1999) were able to obtain a sharp result on the absolute continuity of the positive spectrum for one-dimensional Schrödinger operators using a trace formula by Zaharov and Faddeev (1971); see also Killip and Simon (2009). Such trace formulas provide identities between characteristics of the spectrum (eigenvalues and scattering data) and some functionals involving electric potentials of Schrödinger operators. In addition, they describe integrals of motions of the Korteweg–De Vries (KdV) equation. It is worth noting that one such trace formula was used by Gardner et al. (1974) to prove a special case of what came to be called the Lieb–Thirring inequalities. Later a version of this trace formula for matrix-valued potentials played an important role in obtaining sharp constants in Lieb–Thirring inequalities for multi-dimensional Schrödinger operators (Laptev and Weidl, 2000b). It is remarkable that some sharp constants in Lieb–Thirring inequalities are related to soliton-type potentials appearing in the theory of the KdV equation.

Spectral inequalities in the special case $\gamma = 0$, which correspond to bounds on the number of negative eigenvalues of Schrödinger operators, have an even longer history. After initial results by Bargmann, Birman, Schwinger, Calogero and others, a systematic investigation was started in the Russian school of M. Birman and M. Solomyak and in the US by B. Simon and E. Lieb. The respective estimates are known as Cwikel–Lieb–Rozenblum (CLR) inequalities (Cwikel, 1977; Lieb, 1976, 1980; Rozenbljum, 1972a, 1976). One of the main motivations for proving such inequalities was to obtain necessary and sufficient conditions on the potentials for admitting Weyl asymptotics. By now, there are at least seven proofs of the CLR bound using rather different tools from mathematical analysis.

It has been more than four decades since the paper of Lieb and Thirring was published, and the theory of Lieb–Thirring inequalities is still a very dynamically developing area of functional analysis and mathematical physics. The main conjecture on the sharp Lieb–Thirring constant for $\gamma = 1$ in dimension three remains open. This area has become a well-established part of analysis that generates beautiful new ideas and that finds applications beyond the spectral theory of Schrödinger operators.

This book is aimed at presenting the current state of the art of some parts of spectral theory of partial differential equations. Our intention was to write a book that covers some new results connected to spectral properties of Schrödinger and Laplace operators that have been obtained during the last few decades. Most of them are focused around our own interests related to Lieb–Thirring inequalities. While writing the text and when giving courses based on preliminary versions of the book, we faced the problem that we needed a lot of material from the general spectral theory of self-adjoint operators in Hilbert spaces and a number of standard functional inequalities. Including all this has substantially increased the size of the book and forced us to postpone some core material for the future. We now hope that the book might be useful not only to our colleagues who are interested in spectral inequalities, but also to students specializing in analysis.

Acknowledgements

We would like to express our appreciation to M. Birman, E. Lieb, B. Simon and M. Solomyak who profoundly influenced our mathematical taste. We have been fortunate to have had the opportunity to collaborate with so many individuals on the topic of Lieb–Thirring and spectral inequalities. We would like to thank our teachers and students, colleagues and friends for all the inspiring discussions.

We owe a great debt to J. Peteranderl who read a preliminary version of this book line by line and found countless minor and major inaccuracies. For valuable remarks, suggestions and corrections we are very grateful to C. Dietze, F. Gesztesy, A. Ilyin, V. Kußmaul, S. Larson, M. Lewin, E. Lieb, G. Rozenblum, L. Schimmer and H. Siedentop. Our special thanks go to E. Lieb and B. Simon for their interest in this book and their constant encouragement.

We are grateful to David Tranah and the team at CUP for their assistance when publishing the book.

Parts of this book were written at Caltech, Imperial College London, KTH Stockholm, LMU Munich, Mittag–Leffler Institute, Oberwolfach Research Institute, Stuttgart University, and Tsinghua University, and we would like to thank all these institutions for their hospitality.

Partial support through US National Science Foundation grants PHY-1068285, PHY-1347399, DMS-1363432 and DMS-1954995 and through the Deutsche Forschungsgemeinschaft EXC-2111-390814868 (R.L.F.) is acknowledged.

Munich, London, Stuttgart,
August 2022

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