Overview

The aim of this book is to present qualitative and quantitative results on the discrete spectrum of Schrödinger operators and on Laplacians in Euclidean domains that have appeared in research papers during the last five decades.

If V is a real-valued function on $\mathbb{R}^d$, $d \geq 1$, that is sufficiently regular and tends, at least in some averaged sense, to zero at infinity (we will be more precise about these assumptions later in this book), then the spectrum of the Schrödinger operator $-\Delta - V$ in $L^2(\mathbb{R}^d)$ can be divided into the essential spectrum, which is equal to $[0, \infty)$, and the discrete spectrum, which consists of an at most countable number of negative eigenvalues with finite multiplicities. For general V, these eigenvalues cannot be computed explicitly and, motivated by applications, one is interested in qualitative and quantitative information on their distribution.

A basic result on the negative eigenvalues is Weyl's formula about their asymptotic distribution in the strong coupling limit. More precisely, we consider the family of Schrödinger operators $-\Delta - \alpha V$ with a parameter $\alpha > 0$ and denote their negative eigenvalues, in non-decreasing order and repeated according to multiplicities, by $-E_j(\alpha)$. Then Weyl's law states that, under some assumptions on V and for any $\gamma \geq 0$,

$$\lim_{\alpha \to \infty} \alpha^{-\gamma - d/2} \sum_j E_j(\alpha)^\gamma = L^{\mathrm{cl}}_{\gamma,d} \int_{\mathbb{R}^d} V(x)_+^{\gamma + d/2} \, dx \,,$$

where, for $\gamma = 0$, the sum is interpreted as the number of negative eigenvalues of $-\Delta - \alpha V$. Moreover, we wrote $t_+ := \max\{t, 0\}$. The constant $L^{\mathrm{cl}}_{\gamma,d}$ appearing in this formula is explicit and has an interpretation in terms of a classical phase space integral.

In many applications one also needs, in addition to the asymptotic statement of Weyl's law, non-asymptotic bounds that capture the correct order of magnitude in the asymptotic regime, even if they do not reproduce the correct asymptotic constant. Thus we are interested in bounds of the form

$$\alpha^{-\gamma-d/2} \sum_j E_j(\alpha)^\gamma \le L_{\gamma,d} \int_{\mathbb{R}^d} V(x)_+^{\gamma+d/2}\, dx\,,$$

with a constant $L_{\gamma,d}$ that may depend on γ and d, but is independent of V. Note that it presents no loss of generality to take $\alpha = 1$ in this inequality. The validity of this bound for $\gamma > 1/2$ in $d = 1$ and for $\gamma > 0$ in $d \ge 2$ is a celebrated result of Lieb and Thirring. The case $\gamma = 0$ in $d \ge 3$ is due to Cwikel, Lieb and Rozenblum and the case $\gamma = 1/2$ in $d = 1$ is due to Weidl. This completely settles the validity of this inequality.

Ideally, one would like to know the optimal values of the constants $L_{\gamma,d}$. Some of them are known, but the sharp value of the most important constant $L_{1,3}$, namely for the sum of the negative eigenvalues of the Schrödinger operator in three dimensions, is not. The (still open) Lieb–Thirring conjecture states that the constant $L_{1,3}$ coincides with the constant $L^{\mathrm{cl}}_{1,3}$ that appears in Weyl's asymptotic formula.

In this book, we describe different techniques that establish spectral inequalities of semiclassical type and Weyl's asymptotics for Schrödinger and Laplace operators.

Structure of this book

This book is divided into three parts. The first one contains background material on the spectral theory of linear operators in Hilbert spaces and on Sobolev space theory. Our selection of material here is far from comprehensive and it is guided by what is needed in the later parts of the book. Our presentation in this first part is at times fast-paced and with hardly any illustrative examples, since we expect that most of the readers will have seen at least the basics of the material treated here. More advanced readers may wish to skip Part One and only return to it when some specific technical results in the later parts are needed.

The second part of the book contains the basics of the theory of Laplace and Schrödinger operators, with a special emphasis on their discrete spectra. We do not require any previous acquaintance with these operators and develop the theory starting from what is recalled in Part One. In particular, we will see a close connection between spectral inequalities and Sobolev-type inequalities. In Part Two we prove, among other things, Weyl's asymptotic formula mentioned above and versions of the Cwikel–Lieb–Rozenblum and Lieb–Thirring inequalities, without paying too much attention to the constants.

The third part of the book is devoted to the quest for the sharp constants in the Cwikel–Lieb–Rozenblum and Lieb–Thirring inequalities. This part is more specific and sometimes more technical than the previous two, but, in our opinion, contains some beautiful ideas. In Part Three we prove all the sharp Lieb–Thirring inequalities that are presently known and we also give the proofs of the currently best known results. We hope that this part, besides giving a snapshot of the state of the art, serves the purpose of inviting new researchers to this exciting area of research.

Since every chapter has an introduction describing its content, here we only briefly present the structure of the book.

Chapter 1 contains standard material regarding self-adjoint operators in Hilbert spaces, including the spectral theorem, the variational characterization of eigenvalues and the important relation between lower semibounded quadratic forms and lower semibounded self-adjoint operators. We also discuss the Birman–Schwinger principle, which allows us to reduce problems of counting eigenvalues in the discrete spectrum of unbounded operators to the study of the spectrum of compact operators.

In Chapter 2 we review some material from Sobolev space theory. In particular, we prove some important functional inequalities such as the Sobolev, Gagliardo–Nirenberg, Friedrichs, Poincaré, and Hardy inequalities. As we mentioned before, these inequalities are major tools in the study of spectral inequalities.

In Chapter 3 we introduce the Laplacian with Dirichlet or Neumann boundary conditions on an open set in Euclidean space as a self-adjoint operator. We prove, among other things, Weyl's formula for the asymptotic distribution of its eigenvalues and we present some (sharp) spectral inequalities due to Pólya, Berezin, Li, and Yau.

Chapter 4 is devoted to semibounded Schrödinger operators. After covering explicitly solvable models, we prove Weyl's asymptotic formula, Cwikel–Lieb–Rozenblum and Lieb–Thirring inequalities.

In Chapter 5 we begin our discussion of sharp constants in the latter. We include some general facts about these constants and mention the original conjecture due to Lieb and Thirring. Then we proceed to the proof of two cases of sharp inequalities in one dimension.

In Chapter 6 we derive Lieb–Thirring inequalities with the sharp, semiclassical constant in higher dimension for $\gamma \geq 3/2$. This is achieved by the so-called lifting argument with respect to dimension. The input for this argument is a one-dimensional Lieb–Thirring inequality with matrix-valued potentials, for which we give two different proofs.

In Chapter 7 we collect some additional results concerning sharp Lieb–

Thirring inequalities, including a sharp version of the CLR inequality for radial potentials in dimension 4, as well as counterexamples to parts of the original Lieb–Thirring conjecture.

In Chapter 8 we use again the lifting argument in order to prove the so-far best known constants in Lieb–Thirring inequalities in cases where the sharp constants are not known. We end with a section summarizing the most important results discussed in this book.

Finally, we would like to mention that, while some of the proofs in this book are new and have not been published before, most of them have previously appeared in the literature. We provide references in the comments section at the end of each chapter. We use these sections also to give references to additional, related results and, sometimes, to discuss particularly important results that we did not include in the main text.

Notation and conventions

We use the convention $\mathbb{N} = \{1, 2, 3, \ldots\}$. By $\mathbb{R}$ and $\mathbb{C}$ we denote the real and complex numbers and by $\mathbb{R}^d$ and $\mathbb{C}^d$ their d-fold Cartesian products. Typically, we write $x \in \mathbb{R}^d$ as $(x_1, \ldots, x_d)$ with $x_1, \ldots, x_d \in \mathbb{R}$. The symbol $|\cdot|$ denotes the Euclidean norm and, for $a \in \mathbb{R}^d$ and $r > 0$, we write

$$B_r(a) := \{x \in \mathbb{R}^d : |x - a| < r\}$$

for the open ball of radius r centered at a. Moreover, $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$.

We use the symbols $\subset$ and $\supset$ in the non-strict sense; that is, allowing for the case of equality. Strict inclusions are denoted by $\subsetneq$ and $\supsetneq$.

We use the terms ‘positive’ and ‘negative’ in the strict sense; for instance, a real number is positive if it is non-negative and not zero. Similarly, for a function the terms ‘increasing’ and ‘decreasing’ are used in the strong sense; for instance, a function is increasing if it is non-decreasing and not constant on any non-trivial subinterval. Sometimes, for special emphasis, we write ‘strictly positive’ or ‘strictly increasing’ instead of ‘positive’ and ‘increasing’.

Other notations will be introduced as we proceed; see also the index.