Meromorphic Dynamics

Volume II

This text, the second of two volumes, builds on the foundational material on ergodic theory and geometric measure theory provided in Volume I, and applies all the techniques discussed to describe the beautiful and rich dynamics of elliptic functions. The text begins with an introduction to topological dynamics of transcendental meromorphic functions before progressing to elliptic functions, discussing at length their classical properties, measurable dynamics, and fractal geometry. The authors then look in depth at compactly nonrecurrent elliptic functions. Much of this material is appearing for the first time in book or paper form. Both senior and junior researchers working in ergodic theory and dynamical systems will appreciate what is sure to be an indispensable reference.

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Meromorphic Dynamics

Elliptic Functions with an Introduction to the Dynamics of Meromorphic Functions

VOLUME II

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CAMBRIDGE
UNIVERSITY PRESS

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UNIVERSITY PRESS

Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
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103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment,
a department of the University of Cambridge.

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www.cambridge.org
Information on this title: www.cambridge.org/9781009215978

DOI: 10.1017/9781009215985

First published 2023

Printed in the United Kingdom by TJ Books Limited, Padstow Cornwall

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication data
Names: Kotus, Janina, author. | Urbański, Mariusz. author.
Title: Meromorphic dynamics / Janina Kotus, Warsaw University of
Technology, Mariusz Urbański, University of North Texas.
Description: First edition. | Cambridge, United Kingdom ; New York, NY, USA:
Cambridge University Press, 2023. |
Series: Nmm new mathematical monographs | Includes bibliographical references and index. |
Contents: Volume 1. abstract ergodic theory, geometry, graph directed Markov systems,
and conformal measures – Volume 2. Elliptic functions with an introduction to
the dynamics of meromorphic functions.
Identifiers: LCCN 2022031269 (print) | LCCN 2022031270 (ebook) |
ISBN 9781009216050 (set ; hardback) | ISBN 9781009215916
(volume 1 ; hardback) | ISBN 9781009215978 (volume 2 ; hardback) |
ISBN 9781009215930 (volume 1 ; epub) | ISBN 9781009215985 (volume 2 ; epub)
Subjects: LCSH: Functions, Meromorphic.
Classification: LCC QA331 .K747 2023 (print) | LCC QA331 (ebook) |
DDC 515/.982–dc23/eng20221102
LC record available at https://lccn.loc.gov/2022031269
LC ebook record available at https://lccn.loc.gov/2022031270

ISBN – 2 Volume Set 978-1-009-21605-0 Hardback
ISBN – Volume I 978-1-009-21591-6 Hardback
ISBN – Volume II 978-1-009-21597-8 Hardback

Janina Kotus dedicates this book to the memory of her sister Barbara.
Mariusz Urbański dedicates the book to his family.

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CAMBRIDGE
Cambridge University Press & Assessment
978-1-009-21597-8 — Meromorphic Dynamics
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Volume I

Preface

The ultimate goal of our book is to present a unified approach to the dynamics, ergodic theory, and geometry of elliptic functions mapping the complex plane $\mathbb{C}$ onto the Riemann sphere $\widehat{\mathbb{C}}$. We consider elliptic functions as the most regular class of transcendental meromorphic functions. Poles, infinitely many of them, form an essential feature of such functions, but the set of critical values is finite and an elliptic function is "the same" in all of its fundamental regions. In a sense, this is the class of transcendental meromorphic functions whose resemblance to rational functions on the Riemann sphere is the largest. This similarity is important since the class of rational functions has been, from the dynamical point of view, extensively investigated since the pioneering works of Pierre Fatou [**Fat1**] and Gaston Julia [**Ju**]; also see the excellent historical accounts in [**Al**] and [**AIR**] on the early days of holomorphic dynamics. This similarity can be, and frequently was, a good source of motivation and guidance for us when we were dealing with elliptic functions. On the other hand, the differences are striking in many respects, including topological dynamics, measurable dynamics, fractal geometry of Julia sets, and more. We will touch on them in the course of this Preface. We would just like to stress here that elliptic functions belong to the class of transcendental meromorphic functions and posseses many dynamical and geometric features that are characteristic for this class.

Indeed, the study of iteration of transcendental meromorphic functions, more precisely of transcendental entire functions, began with the pioneering works of Pierre Fatou ([**Fat2**] and [**Fat3**]). Then for about two decades, beginning with paper [**Ba1**], I.N. Baker was actually the sole mathematician dealing with the dynamics of transcendental entire functions. It was Janina Kotus's idea to study the iteration of meromorphic functions despite the existence of poles that, in general, cause the second iterate to not be defined everywhere; more precisely, because it has finite essential singularities, thus, it is not a meromorphic function defined on the complex plane. This is the phenomenon that had been deterring mathematicians from dealing with the iteration of general meromorphic functions. Janina was not afraid and as a result, to our knowledge, the first, and quite systematic, account of the dynamics of general transcendental meromorphic functions was set up in a series of works by I.N. Baker, J. Kotus, and Y. Lü ([**BKL1**]–[**BKL4**]). Since then this subfield of dynamical systems has been flourishing. Of great importance for the development of this subject was the excellent expository article by Walter Bergweiler [**Ber1**]. We would also like to mention an early paper by Alexander Eremenko and Misha Lyubich [**EL2**], who introduced and studied class $\mathcal{B}$ of transcendental entire functions. The definition of class $\mathcal{B}$ literally extends to the class of general meromorphic functions and plays an important role in this field too.

The area of transcendental meromorphic dynamical systems is also a beautiful and vast field for investigations of the measurable dynamics they generate and the fractal geometry of their Julia sets and their significant subsets. Here, the early papers by Misha Lyubich [**Ly**] and Mary Rees [**Re**] on measurable dynamics come to the fore. The study of the fractal geometry of Julia sets of meromorphic functions began with two papers by Gwyneth Stallard ([**Sta1**] and [**Sta2**]) and has been continued ever since by her, Krzysztof Barański, Walter Bergweiler, Bogusia Karpińska, Volker Mayer, Phil Rippon, Lasse Rempe-Gillen, Anna Zdunik, the authors of this book, and many more mathematicians, obtaining interesting and sophisticated results; we are not able to list all of them here. We would like, however, to mention the early paper by Anna Zdunik and Mariusz Urbański [**UZ1**], where the concept of conformal measures was adapted for and used in transcendental dynamics, and also Hausdorff and packing measures of (radial) Julia sets were studied in detail.

We would also like to single out a paper by Krzysztof Barański [**Ba**], who initiated the use of thermodynamic formalism in transcendental meromorphic dynamics. More papers using and developing thermodynamic formalism then followed, among them those by Janina Kotus [**KU1**], Anna Zdunik [**UZ2**], and Volker Mayer ([**MyU4**] and [**MyU5**]), all written jointly with Mariusz Urbański. As a source of detailed information on many aspects of the use of this method in meromorphic dynamics, we recommend the expository article by Volker Mayer and Mariusz Urbański [**MyU6**].

As already stated at the beginning of this Preface, we ultimately focus in this book on the dynamics, ergodic theory, and geometry of elliptic functions. To our knowledge, the first dynamical result specific to elliptic functions appeared in Janina Kotus's 1995 paper [**Ko3**]. It gave a good lower bound for the Hausdorff dimensions of the Julia setd of elliptic functions. Its was refined, using the theory of countable alphabet conformal iterated function systems, in [**KU3**]. A later paper [**MyU1**] presents a form of thermodynamic formalism for elliptic functions. A quite long series of papers by Jane Hawkins and her collaborators, studying the dynamics and geometry of Weierstrass $\wp$-functions, began in 2002 with [**HK1**]. Our book stems from, has been motivated by, and largely develops our 2004 paper [**KU4**]

We devote the first chapter of the second volume to a rather short and compressed, albeit with proofs, introduction to the topological dynamics of transcendental meromorphic functions. We then move on to elliptic functions, giving first some short, but with proofs, expositions of classical properties of such functions, and then we deal with their measurable dynamics and fractal geometry. We single out several dynamically significant subclasses of elliptic functions, primarily nonrecurrent, compactly nonrecurrent, subexpanding, and parabolic. We devote one long chapter to describing examples of elliptic functions with various properties of their dynamics, Fatou Connected Components, and the geometry of Julia sets.

Our approach to measurable dynamics and the fractal geometry of elliptic functions is founded on the concept of the Sullivan conformal measures. We prove their existence for all (nonconstant) elliptic functions and provide several characterizations, in dynamical terms, of the minimal exponent for which these measures exist. By using the method of conformal iterated function systems with a countable alphabet and essentially reproducing the proof from [**KU3**], we provide a simple lower bound, expressed in terms of orders of poles, of the Hausdorff dimensions of Julia sets of all elliptic functions. It follows from this estimate that such Hausdorff dimensions are always strictly larger than 1. We also provide a closed exact formula for the Hausdorff dimension of the set of points escaping to ∞. We then deal with the class of nonrecurrent elliptic functions $f : \mathbb{C} \to \widehat{\mathbb{C}}$ and their subclasses such as compactly nonrecurrent, subexpanding, and parabolic ones. This is the ultimate object of our interests in the book, especially in Volume II. We would like to add that we do not give separate attention to hyperbolic/expanding elliptic functions; none of them allow critical points or rationally indifferent periodic points in their Julia sets. Indeed, doing this would take up a lot of pages and preparations, and a good account of the thermodynamic formalism of quite general classes of meromorphic functions is given in [**MyU4**] and [**MyU5**], as well as [**MyU3**] and [**MyU6**]. In this book, we focus on elliptic functions that may have critical points and rationally indifferent periodic points in the Julia sets; we do allow them, and our main objective is to deal with and study the various phenomena that they cause.

Our presentation of the theory of nonrecurrent elliptic functions is based on an appropriate version, which we prove, of Mañé's Theorem, which roughly speadking asserts that the connected components of all inverse images of all orders of all sufficiently small sets remain small. Having a Sullivan conformal measure m with a minimal exponent, we prove its uniqueness and atomlessness for compactly nonrecurrent regular elliptic functions.

Next, we prove the existence and uniqueness (up to a nonzero multiplicative factor) of a σ-finite invariant measure μ that is absolutely continuous with respect to the conformal measure m. We prove its ergodicity and conservativity. Restricting our attention to the classes of subexpanding and parabolic functions, in fact to some natural subclasses of them, we prove much more refined stochastic properties of the dynamical system (f, μ). Our approach here stems from and largely develops the methods developed in papers [**ADU**], [**DU4**], [**U3**], and [**U4**]. It is, however, significantly enlarged and enriched, via the powerful tool of nice sets, by the methods of countable alphabet conformal iterated function systems and by graph directed Markov systems as developed and presented in [**MU1**] and [**MU2**]. These in turn are substantially based on the theory of countable alphabet thermodynamic formalism developed in [**MU5**] and [**MU2**]. When dealing with subexpanding functions, especially with the exponential shrinking property, the paper by Przytycki and Rivera-Letelier [**PR**] was also very useful. The finer stochastic properties mentioned above are primarily the exponential decay of correlations, the Central Limit Theorem, and the Law of the Iterated Logarithm in the case of subexpanding elliptic functions. In the case of parabolic elliptic functions for which the invariant measure μ is finite, we prove the Central Limit Theorem. All of these are achieved with the help of the Lai-Sang Young tower techniques from [**LSY3**]. In the case of parabolic elliptic functions for which the invariant measure μ is infinite, we prove an appropriate version of the Darling–Kac Theorem, establishing the strong convergence of weighted Birkhoff averages to Mittag–Leffler distributions.

Last, we would like to mention finer fractal geometry. For both subexpanding and parabolic elliptic functions, we give a complete description and characterization of conformal measures and Hausdorff and packing measures of Julia sets. Because the Hausdorff dimension of the Julia set of an elliptic function is strictly larger than 1, this picture is even simpler than for the subexpanding, parabolic, and nonrecurrent rational functions given in [**DU4**], [**DU5**], and [**U3**].

In order to comprehensively cover the dynamics and geometry of elliptic functions described above, we extensive large preparations. This is primarily done in the first volume of the book, which consists of two parts: Part I, "Ergodic Theory and Geometric Measures" and Part II, "Complex Analysis, Conformal Measures, and Graph Directed Markov Systems." We intend our book to be as self-contained as possible and we use essentially all the major results from Volume I in Volume II when dealing with dynamics, ergodic theory, and the geometry of elliptic functions.

Our book can thus be treated not only as a fairly comprehensive account of dynamics, ergodic theory, and the fractal geometry of elliptic functions but also as a reference book (with proofs) for many results of geometric measure theory, finite and infinite abstract ergodic theory, Young towers, measure-theoretic Kolmogorov–Sinai entropy, thermodynamic formalism, geometric function theory (in particular the Koebe Distortion Theorems and Riemann–Hurwitz Formulas), various kinds of conformal measures, conformal graph directed Markov systems and iterated function systems, the classical theory of elliptic functions, and the topological dynamics of transcendental meromorphic functions.

The material contained in Volume I of this book, after being substantially processed, collects, with virtually all proofs, the results that are essentially known and have been published. However, Chapter 5 contains material on infinite ergodic theory that, to the best or our knowledge, has not been included, with full proofs, in any prior book. Also, Chapter 12, which treats nice sets, is strongly processed and goes, in many respects, far beyond the existing knowledge and use of nice sets and nice families in conformal dynamics. The need for such far-reaching extensions of this method comes from the need for its applications to parabolic elliptic functions.

Most of the material at the end of the second volume of this book is actually new, although we borrow, use, and apply much from the previous results, methods, and techniques. Indeed, Chapter 17, except its last two sections, is entirely new. Also, to the best of our knowledge, Section 19.4 is purely original, providing large classes of a variety of simple examples of various kinds of dynamically elliptic functions. Part VI is, indeed, entirely original.

Acknowledgments

We are very indebted to Jane Hawkins, who gave for us the four images of Julia sets of various Weierstrass elliptic $\wp$-functions that are included in Chapter 19 of Volume II of this book. We also thank her very much for fruitful discussions about the dynamics and Julia sets of Weierstrass elliptic $\wp$-functions. Mariusz Urbański would like to thank William Cherry for helpful discussions about normal families and Montel's Theorems. Last, but not least, we want to express our gratitude to the referees of this book, whose numerous valuable remarks, comments, and suggestions prompted us to improve the content of the book and the quality of its exposition. The research of Janina Kotus was supported by the National Science Center, Poland, decision no. DEC-2019/33/B/ST1/00275. The reseach of Mariusz Urbański was supported in part by the NSF grant DMS 1361677.

Introduction

In this introduction, we describe in detail the content of the second volume of the book, simultaneously highlighting its sources and the interconnections between various fragments of the book. Most of this volume is devoted to a direct exploration of the dynamics and geometry of elliptic functions. Indeed, all but one, the first, parts of the volume, i.e., Parts IV–VI, do this.

The first part of this volume, i.e., Part III, "Topological Dynamics of Meromorphic Functions," is devoted to the iteration of arbitrary meromorphic functions. Indeed, it provides a relatively short and condensed account of the topological dynamics of almost all meromorphic functions with an emphasis on Fatou domains, including a detailed account of Baker domains that are exclusive for transcendental functions and do not occur for rational functions. We actually do this for all meromorphic functions, occasionally restricting our attention to the class of transcendental meromorphic functions all of whose prepoles (that include poles) form an infinite set. Essentially, all results of this part are supplied with full proofs. In particular, we provide a complete proof of Fatou's classification of Fatou Periodic Components. We do a thorough analysis of the singular set of the inverse of a meromorphic function and all its iterates; in particular, we study at length asymptotic values and their relations to transcendental tracts. We analyze the structure of these components and the structure of their boundaries in greater detail. In particular, we provide a very detailed qualitative and quantitative description of the local behavior of locally and globally defined analytic functions around rationally indifferent periodic points and of the structure of corresponding Leau–Fatou flower petals, including the Fatou Flower Petal Theorem. Such an analysis will turn out to be an indispensable tool in the last three sections of Chapter 22 in Part VI, where we deal with the ergodic theory of parabolic elliptic functions. We also distinguish Speiser class $\mathcal{S}$ and Eremenko–Lyubich class $\mathcal{B}$ of meromorphic functions, which play a seminal role in the recent development of the theory of iteration of transcendental meromorphic functions, proving their fundamental properties, which include some structural theorems about their Fatou components such as no existence of Baker domains and wandering domains (the Sullivan Nonwandering Theorem) for class $\mathcal{S}$. The proof of the latter theorem, because of its length and high technicality, is however relegated to Appendix B.

To the best of our knowledge, there is no systematic book account of the topological dynamics of transcendental meromorphic functions. Some results, with and without proofs, can be found in [**BKL1**]–[**BKL4**] and in [**Ber1**]. As we have already said, essentially all results in Part III of our book are supplied with proofs.

In Part IV, we move on to elliptic functions and stay with them until the end of the book. The first chapter of this part, i.e., Chapter 16, which is interesting on its own, is devoted to presenting an account of the classical theory of elliptic functions. Almost no dynamics is involved here. We will actually not need this chapter anywhere else in the book except in Chapter 19, where we provide many examples of elliptic functions, including mainly but, we want to emphasize this, not only Weierstrass $\wp$ functions. Here, we primarily follow the classical books [**Du**] and [**JS**]. We would also like to draw the reader's attention to the books [**AE**] and [**La**].

Throughout the whole of Chapter 17, we deal with general nonconstant elliptic functions, i.e., we impose no constraints on a given nonconstant elliptic function. We first systematically deal with forward and, more importantly, backward images of open connected sets, especially those with connected components of the latter. We mean to consider such images under all iterates f^n, $n \geq 1$, of a given elliptic function f. We do a thorough analysis of the singular set of the inverse of a meromorphic function and all its iterates; in particular, we study at length asymptotic values and their relations to transcendental tracts. We also provide sufficient conditions for the restrictions of iterates f^n to such components to be proper or covering maps. Both of these methods, the latter allowing the use of the machinery of Section 8.6 from Volume I, are our primary tools to study the structure of connected component backward images of open connected sets. In particular, they prove the existence of holomorphic inverse branches if "there are no critical points." Holomorphic inverse branches will be one of the most common tools used throughout the rest of the book. We then apply these results to study images and backward images of connected components of the Fatou set.

Section 17.2 continues this theme, providing some structural theorems about Fatou and Julia sets of elliptic functions. Some of these are the immediate consequences of the results obtained in Part III, "Topological Dynamics of

Meromorphic Functions," once we observed that each elliptic function belongs to Speiser class $\mathcal{S}$, while others are more technically complicated.

The rest of Chapter 17 is actually devoted to analyzing in greater detail the fractal properties of any nonconstant elliptic function. Following the paper [**KU3**], by associating with a given elliptic function an infinite alphabet conformal iterated function system, and heavily utilizing its θ number, we provide a strong, somewhat surprising, lower bound for the Hausdorff dimension of the Julia sets of all nonconstant elliptic functions. In particular, this estimate shows that the Hausdorff dimension of the Julia sets of any nonconstant elliptic function is strictly larger than 1. We also provide a simple closed formula for the Hausdorff dimension of the set of points escaping to infinity under iteration of an elliptic function. In the last section of this chapter, we prove that no conformal measure of an elliptic function charges the set of escaping points. However, the central focus of this chapter is Section 17.6, where we prove the existence of the Sullivan conformal measures with a minimal exponent for all elliptic functions and we characterize the value of this exponent in several dynamically significant ways. Section 17.6 depends on the preparatory work in Sections 17.4 and 17.5, which are also interesting on their own. It also heavily depends on Chapter 10 in the first volume.

In Part V, "Compactly Nonrecurrent Elliptic Functions: First Outlook," we define the class of nonrecurrent and, more notably, the class of compactly nonrecurrent elliptic functions. This is the class of elliptic functions that will be dealt with by us from the moment compactly nonrecurrent elliptic functions are defined until the end of the book. Its history goes back to the papers [**U3**], [**U4**], and [**KU4**]. One should also mention the paper [**CJY**]. Similarly to all the papers that our treatment of nonrecurrent elliptic functions is based on, the fact that this is possible at all is due to an appropriate version of the breakthrough Mañé's Theorem that was proven in [**M1**] in the context of rational functions. Without Mañé's Theorem, such treatment would not be possible. In our setting of elliptic functions, this is Theorem 18.1.6. The first section of Chapter 18 is entirely devoted to proving this theorem, its first most fundamental consequences, and some other results surrounding it. The next two sections of this chapter, also relying on Mañé's Theorem, provide us with further refined technical tools to study the structure of Julia sets and holomorphic inverse branches.

The last section of this chapter, i.e., Section 18.4, has a somewhat different character. It systematically defines and describes various subclasses of the, mainly compactly nonrecurrent, elliptic functions that we will be dealing with in Part VI of the book. Mostly, but not exclusively, these classes of elliptic functions are defined in terms of how strongly expanding these functions are. We would like to add that while in the theory of rational functions such classes pop up in a natural and fairly obvious way, e.g., metric and topological definitions of expanding rational functions describe the same class of functions, in the theory of iteration of transcendental meromorphic functions such a classification is by no means obvious as the topological and metric analogs of rational function concepts do not usually coincide and the definitions of expanding, hyperbolic, topologically hyperbolic, subhyperbolic, etc. functions vary from author to author. Our definitions seem to us to be quite natural and fit well with our purpose of the detailed investigation of the dynamical and geometric properties of the elliptic functions that they define. In this section, we also define the class of regular elliptic functions. The condition defining them is quite simple but, although very frequently holding, it does not look natural. It is, in fact, tailor-made for the proof of the possibly (in a sense) richest properties of the Sullivan conformal measures obtained in Section 20.3 to go through.

The purpose of Chapter 19 is to provide examples of elliptic functions with prescribed properties of the orbits of critical points (and values). We primarily focus on constructing examples of the various classes of compactly nonrecurrent elliptic functions discerned in Section 18.4. All these examples are either Weierstrass $\wp_\Lambda$ elliptic functions or their modifications. The dynamics of such functions depends heavily on the lattice Λ and varies drastically from Λ to Λ.

The first three sections of this chapter have a preparatory character and, respectively, describe the basic dynamical and geometric properties of all Weierstrass $\wp_\Lambda$ elliptic functions generated by square and triangular lattices Λ.

In Section 19.4, we provide simple constructions of many classes of elliptic functions discerned in Section 18.4. We essentially cover all of them. All these examples stem from Weierstrass $\wp$ functions.

We then, starting with Section 19.5, also provide some different, interesting on their own, and historically first examples of various kinds of Weierstrass $\wp$ elliptic functions and their modifications. These come from the series of papers [**HK1**], [**HK2**], [**HK3**], [**HKK**], and [**HL**] by Hawkins and her collaborators.

Part VI, "Compactly Nonrecurrent Elliptic Functions: Fractal Geometry, Stochastic Properties, and Rigidity," is entirely devoted to getting the dynamical, geometric/fractal, and stochastic properties of dynamical systems generated by compactly nonrecurrent elliptic functions, primarily subexpanding and parabolic ones.

In Chapter 20, we use the fruits of the existence of the Sullivan conformal measures with a minimal exponent proven in Section 17.6 and its dynamical characterizations obtained therein. Having compact nonrecurrence, we are able to prove in the first section of this chapter that this minimal exponent is equal to the Hausdorff dimension $\mathrm{HD}(J(f))$ of the Julia set $J(f)$, which we always denote by h. We also obtain in this section strong restrictions on the possible locations of atoms of such conformal measures.

Section 20.3, the last section in Chapter 20, is a culmination of our work on the Sullivan conformal measures for elliptic functions treated on their own. There, and from then onward, we assume that our compactly nonrecurrent elliptic function is regular, which is the concept introduced in Section 18.4. For this class of elliptic functions, we prove the uniqueness and atomlessness of h-conformal measures along with their first fundamental stochastic properties such as ergodicity and conservativity.

The results of Chapter 20 are not, however, the last word on the Sullivan conformal measures. Left alone, these measures would be a kind of curiosity that is perhaps only worthy of shrugging shoulders and raised eyebrows. Their true power, meaning, and importance come from their geometric characterizations, and, more accurately, from their usefulness – one could even say indispensability – for understanding geometric measures on Julia sets, i.e., their Hausdorff and packing h-dimensional measures, where, we recall, $h = \mathrm{HD}(J(f))$. This is fully achieved in Chapter 21 for compactly nonrecurrent regular elliptic functions. Having said this, Chapter 21 can be viewed from two perspectives. The first is that we provide therein a geometrical characterization of the h-conformal measure m_h, which, with the absence of parabolic points, turns out to be a normalized packing measure; the second is that we give a complete description of geometric, Hausdorff, and packing measures of the Julia sets $J(f)$. All of this is contained in Theorem 21.0.1, which gives a simple clear picture. Because of the fact that the Hausdorff dimension of the Julia set of an elliptic function is strictly larger than 1, this picture is even simpler than for nonrecurrent rational functions of [**U3**]; see also [**DU5**].

Throughout the whole of Chapter 22, $f: \mathbb{C} \to \widehat{\mathbb{C}}$ is assumed to be a compactly nonrecurrent regular elliptic function. This chapter is, in a sense, the core of our book. Taking the fruits of what has been done in all previous chapters, we prove in Chapter 22 the existence and uniqueness, up to a multiplicative constant, of a σ-finite f-invariant measure μ_h equivalent to the h- conformal measure m_h. Furthermore, still heavily relying on what has been done in all previous chapters, particularly on conformal graph directed Markov systems, nice sets, first return map techniques, and Young towers, we provide here a systematic account of the ergodic and refined stochastic properties of the dynamical system (f, μ_h) generated by such subclasses of compactly nonrecurrent regular elliptic functions as normal subexpanding elliptic functions of finite character and parabolic elliptic functions. By stochastic properties, we mean here the exponential decay of correlations, the Central Limit Theorem, the Law of the Iterated Logarithm for subexpanding functions, the Central Limit Theorem for those parabolic elliptic functions for which the invariant measure μ_h is finite (probabilistic after normalization), and an appropriate version of the Darling–Kac Theorem that establishes the strong convergence of weighted Birkhoff averages to Mittag–Leffler distributions for those parabolic elliptic functions for which the invariant measure μ_h is infinite.

In Chapter 23, the last actual chapter of the book, we deal with the problem of dynamical rigidity of compactly nonrecurrent regular elliptic functions. The issue at stake is whether a weak conjugacy such as a Lipschitz one on Julia sets can be promoted to a much better one such as an affine conjugacy on the whole complex plane $\mathbb{C}$. This topic has a long history and goes back at least to the seminal paper [**Su4**] by Sullivan, who treated, among others, the dynamical rigidity of conformal expanding repellers. Its systematical account can be found in [**PU2**]. A large variety, in many contexts, both smooth and conformal, of dynamical rigidity theorems have been proved. The literature abounds.

Our approach in this chapter stems from the original article by Sullivan [**Su4**]. It is also influenced by [**PU1**], where the case of tame rational functions has actually been done, and [**SU**], where the equivalence of statements (1) and (4) of Theorem 23.0.1 was established for all tame transcendental meromorphic functions. Being tame means that the closure of the postsingular set does not contain the whole Julia set; in particular, each nonrecurrent elliptic function is tame. We would, however, like to emphasize that, unlike [**SU**], we chose in our book the approach that does not make use of the dynamical rigidity results for conformal iterated function systems proven in [**MPU**].

In Appendix A, “A Quick Review of Some Selected Facts from Complex Analysis of a One Complex Variable,” we collect for the convenience of the reader many basic and fundamental theorems of complex analysis. We provide no proofs, but we give detailed references (quite arbitrarily chosen) where the proofs can be found. We use these theorems throughout the book, frequently without directly referring to them. The content of Appendix B is clear from its title. It stems from the Sullivan breakthrough paper [**Su1**] and follows closely the proof presented in [**BKL4**].