13

Fundamental Properties of Meromorphic Dynamical Systems

In this chapter, we provide a relatively short and condensed account of the topological dynamics of all meromorphic functions, with an emphasis on Julia sets and Fatou domains, including Baker domains that are exclusive for transcendental functions and do not occur for rational functions. We do this for all meromorphic functions. In particular, we provide a complete proof of Fatou's classification of Fatou Periodic Components. We analyze the structure of these components and the structure of their boundaries in greater detail.

We also do a thorough analysis of the singular set of the inverse of a meromorphic function and all its iterates; in particular, we study at length asymptotic values and their relations to transcendental tracts. The results of this analysis will be very frequently used to study the topological structure of connected components (and their boundaries) of Fatou sets in this part of the book and a countless number of times when we move on to dealing with elliptic functions.

To the best of our knowledge, there is no systematic book account of the topological dynamics of transcendental meromorphic functions. Some results, with and without proofs, can be found in [**BKL1**]–[**BKL4**] and in [**Ber1**]. For the iteration of rational functions, the reader may consult [**CaGg**], [**Bea**], [**Ste**], and [**Mil1**]. Essentially, all results in this chapter are supplied with proofs.

13.1 Basic Iteration of Meromorphic Functions

In this section, we define Fatou and Julia sets of meromorphic functions. We also classify all periodic points of such functions. We prove some basic, rather elementary facts about of all of them.

Given a meromorphic function $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ and a set $A \subseteq \widehat{\mathbb{C}}$, the set

$$f^{-1}(A) := \{z \in \mathbb{C} : f(z) \in A\}$$

has the standard meaning. We then define $f^{-0}(A) := A$ and the sets $f^{-n}(A)$, $n \geq 2$, by induction as follows:

$$f^{-n}(A) := f^{-(n-1)}(f^{-1}(A)). \tag{13.1}$$

Then, for every $n \geq 0$, there is a well-defined nth-folded composition function

$$f^n : f^{-n}(\widehat{\mathbb{C}}) \longrightarrow \widehat{\mathbb{C}}$$

and

$$f^{-n}(\widehat{\mathbb{C}}) = \mathbb{C} \setminus \bigcup_{k=1}^{n-1} f^{-k}(\infty) = \widehat{\mathbb{C}} \setminus \bigcup_{k=0}^{n-1} f^{-k}(\infty). \tag{13.2}$$

In particular, whenever we write $f^n(z)$, we assume that $z \in f^{-n}(\widehat{\mathbb{C}})$. Since the function $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is continuous, by using (13.1) and (13.2), along with the fact that complements of countable subsets in $\widehat{\mathbb{C}}$ (and $\mathbb{C}$) are connected, we directly obtain the following.

Observation 13.1.1 Let $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ be a meromorphic function and $n \geq 1$ be an integer. Then

(1) $f^{-n}(\widehat{\mathbb{C}})$ is an open connected subset of $\mathbb{C}$.
(2) The set $\bigcup_{k=1}^{n-1} f^{-k}(\infty)$ is countable and closed in $\mathbb{C}$.
(3) The function $f^n : f^{-n}(\widehat{\mathbb{C}}) \longrightarrow \widehat{\mathbb{C}}$ is meromorphic, in particular continuous.

Consequently,

(4) If $A \subseteq \widehat{\mathbb{C}}$ is an open set, then the set $f^{-n}(A)$ is open in $\mathbb{C}$ (and $\widehat{\mathbb{C}}$ too).
(5) If $A \subseteq \mathbb{C}$ is a closed set, then the set $f^{-n}(A)$ is also closed (in $\mathbb{C}$).

Definition 13.1.2 Let $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ be a meromorphic function. Given a natural number $n \geq 0$, we call the elements of the set $f^{-n}(\infty)$ the prepoles of f of order n.

Note that a pole is just an order 1 prepole and ∞ is the sole prepole of order 0. We shall prove the following slight strengthening of Observation 13.1.1(4).

Lemma 13.1.3 *For every integer $n \geq 0$, the set of accumulation points of $f^{-n}(\infty)$ in $\widehat{\mathbb{C}}$ is contained in*

$$\{\infty\} \cup \bigcup_{k=0}^{n-1} f^{-k}(\infty).$$

Proof Fix $n \in \mathbb{N}$. Let $z \in \mathbb{C}$ be an accumulation of $f^{-n}(\infty)$. Then there exists a sequence $(z_k)_{k=1}^{\infty}$ of mutually distinct elements in the set $f^{-n}(\infty)$ such that

$$z = \lim_{k\to\infty} z_k.$$

Seeking contradiction, suppose that

$$z \notin \bigcup_{k=0}^{n-1} f^{-k}(\infty).$$

Then there exists a neighborhood U of z such that f^n restricted to U is a meromorphic function. But this is a contradiction since f^n has poles at all points z_k, $k \geq 1$, and infinitely many of them belong to U. The proof is complete. ■

We recall that a meromorphic function $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is transcendental if and only if ∞ is its essential singularity.

The most fundamental definitions in this volume of the book are these.

Definition 13.1.4 The Fatou set $F(f)$ of a meromorphic function $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is defined in exactly the same manner as for rational functions: $F(f)$ is the set of all points $z \in \mathbb{C}$ for which all the iterates f^n of f are defined, i.e., $z \in \text{Int}\left(\bigcap_{n=0}^{\infty} f^{-n}(\widehat{\mathbb{C}})\right)$, and form a normal family on some neighborhood of z.

Definition 13.1.5 The *Julia set* $J(f)$ of a meromorphic function $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is defined to be the complement of $F(f)$ in $\widehat{\mathbb{C}}$, i.e., $J(f) = \widehat{\mathbb{C}} \setminus F(f)$.

Thus, the Fatou set $F(f)$ is open while the Julia set $J(f)$ is closed. We adopt the following definition.

Definition 13.1.6 A meromorphic function $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is said to be

(1) entire if and only if $f(\mathbb{C}) \subseteq \mathbb{C}$;
(2) nearly entire if and only if it is either entire or $f^{-1}(\infty)$ is a singleton whose (only) element is an omitted value;
(3) nonnearly entire if and only if it is transcendental and not nearly entire. The class of nonnearly entire functions will be denoted by NN$\mathcal{E}$.

In what follows, we will try to impose on the meromorphic function $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ as weak assumptions as possible. Eventually, we will assume that

$f \in \mathrm{NN}\mathcal{E}$. Although we regret it, we do this for two interrelating reasons. First, since at some point the development and particular proofs for nonnearly entire functions diverge from those that are either entire or rational; second, since ultimately we will deal in this book with elliptic functions and these all are nonnearly entire. In addition, the literature on entire and rational functions is quite rich (see the introduction to this chapter), so we do not feel too guilty.

For all $z \in \widehat{\mathbb{C}}$, we define

$$O^+(z) := \{f^n(z) : n \geq 0\},$$

i.e., the forward orbit of z with the convention, taken here, that $f(\infty) = \infty$, and

$$O^-(z) := \bigcup_{n=0}^{\infty} f^{-n}(z),$$

i.e., the backward orbit of z. We provide here the following characterization of the class of nonnearly entire functions which is a straightforward consequence of Picard's Great Theorem.

Theorem 13.1.7 *If $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is a transcendental meromorphic function, then the following statements are equivalent.*

(1) $f \in \mathrm{NN}\mathcal{E}$.
(2) $f^{-2}(\infty) \neq \emptyset$.
(3) *The set $f^{-2}(\infty)$ is infinite.*
(4) *The set $O^-(\infty)$ is infinite.*

As an immediate consequence of this theorem and Montel's Theorem II, i.e., Theorem 8.1.16, we get the following.

Theorem 13.1.8 *If $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is a nonnearly entire meromorphic function, then*

$$J(f) = \overline{\bigcup_{n=0}^{\infty} f^{-n}(\infty)} \tag{13.3}$$

and

$$F(f) = \mathrm{Int}\left(\bigcap_{n=0}^{\infty} f^{-n}(\widehat{\mathbb{C}})\right). \tag{13.4}$$

We now recall the basic properties of the Fatou set and the Julia set. It can be directly seen from the definitions that the Fatou set $F(f)$ is completely invariant while

$$f^{-1}(J(f)) \subset J(f) \quad \text{and} \quad f(J(f)\backslash\{\infty\}) = J(f). \tag{13.5}$$

We shall prove the following.

Theorem 13.1.9 *If $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is a meromorphic function, then either $J(f) = \widehat{\mathbb{C}}$ or $J(f)$ has an empty interior (is nowhere dense).*

Proof Suppose that $J(f)$ has a nonempty interior. Denote this interior by W. Let $W_\infty = \bigcup_{n=0}^{\infty} f^n(W)$. If $\widehat{\mathbb{C}}\backslash W_\infty$ contains three distinct points, then Montel's Theorem II, i.e., Theorem 8.1.16, yields that the family of iterates $\{f^n|_W\}$ is normal; therefore, $W \subseteq F(f)$. This is a contradiction; thus, $\widehat{\mathbb{C}}\backslash W_\infty$ contains at most two points. Hence, $J(f) \supseteq \overline{W_\infty} = \widehat{\mathbb{C}}$. The proof is complete. ■

We say that a point $z \in \mathbb{C}$ is exceptional if and only if the set $O^-(z)$ is finite. Picard's Great Theorem tells us that a transcendental meromorphic function can have at most two exceptional values. Again, Montel's Theorem II, i.e., Theorem 8.1.16, along with (13.5) imply that if z is not exceptional and $z \in J(f)$, then

$$J(f) = \overline{O^-(z)}. \tag{13.6}$$

We recall that a subset of a topological space is called perfect if and only if it contains no isolated points. We shall prove the following.

Theorem 13.1.10 *If $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is a nonnearly entire meromorphic function, then the Julia set $J(f)$ is perfect.*

Proof Fix $z \in J(f)$ arbitrary. Let U be an open neighborhood of z. As, by Theorem 13.1.7, $O^-(\infty)$ is infinite, we can find three mutually distinct points: $z_1, z_2, z_3 \in O^-(\infty)\backslash O^+(z)$. Since the sequence $\{f^n_{|U}\}$ is not normal, $z_j \in O^+(U)$ for at least one $j \in \{1,2,3\}$. Hence, $O^-(z_j) \cap (U\backslash\{z\}) \neq \emptyset$. As $O^-(z_j) \subseteq J$, this entails that $J \cap (U\backslash\{z\}) \neq \emptyset$. Hence, z is not isolated in J. Thus, $J(f)$ is perfect and the proof is complete. ■

Now we provide the classical classification of periodic points. A point $\xi \in \mathbb{C}$ is called periodic if

$$f^p(\xi) = \xi$$

for some $p \geq 1$. In this case, the number p is called a period of ξ, and the smallest p with this property is called the minimal (or prime) period of ξ. If $p = 1$, then ξ is also called (naturally) a fixed point of f. We denote by

$$\mathrm{Per}(f), \quad \mathrm{Per}_p(f), \quad \text{and} \quad \mathrm{Per}_p^*(f),$$

respectively, the set of all periodic points of f, all periodic points of f of period p, and all periodic points of f of period prime p. If ξ is a periodic point of f of prime period p, then the complex number

$$(f^p)'(\xi)$$

is called the multiplier of ξ. We classify periodic points of f as follows.

Definition 13.1.11 Let ξ be a periodic point of a meromorphic function $f: \mathbb{C} \to \widehat{\mathbb{C}}$ with minimal period $p \geq 1$. The periodic point ξ is called

(1) *attracting*,
(2) *super-attracting* (of course, being super-attracting yields attracting),
(3) *indifferent* (or neutral), or
(4) *repelling*,

respectively, as the modulus of its multiplier is less than 1, equal to 0, equal to 1, or greater than 1.

Definition 13.1.12 Writing the multiplier of an indifferent periodic point ξ in the form $e^{2\pi i\alpha}$ where $0 \leq \alpha < 1$, we say that

(a) ξ is rationally indifferent (parabolic) if α is rational, and
(b) ξ is irrationally indifferent if α is irrational.
(c) If ξ is a rationally indifferent fixed point of f and $f'(\xi) = 1$, then ξ is called a simple rationally indifferent (parabolic) fixed point of f.

If ξ is an attracting periodic point of f with minimal period $p \geq 1$, then, for all sufficiently small $R > 0$, we have that

$$f^p(B(\xi, R)) \subset B\left(\xi, \frac{1 + |(f^p)'(\xi)|}{2} R\right) \subset B(\xi, R).$$

Thus, it follows from Montel's Theorem II, i.e., Theorem 8.1.16, that $B(\xi, R) \subset F(f)$. So, if we define

$$A(\xi) := \left\{z \in \mathbb{C} : \lim_{n\to\infty} f^{pn}(z) = \xi\right\},$$

then $B(\xi, R) \subset A(\xi) \subset F(f)$; furthermore, if we denote by $A^*(\xi)$ the connected component of $A(\xi)$ that contains ξ, then $B(\xi, R) \subset A^*(\xi)$ and

$$A(\xi) = \bigcup_{n=0}^{\infty} f^{-n}(B(\xi, R)) = \bigcup_{n=0}^{\infty} f^{-n}(A^*(\xi)).$$

Since all limit points of iterates of f on $A(\xi)$ are constant functions (with values in $\{\xi, f(\xi), \ldots, f^{p-1}(\xi)\}$), we conclude that no point on the boundary of $A^*(\xi)$ may belong to the Fatou set $F(f)$. Thus, the set $A^*(\xi)$ is a connected component of the Fatou set $F(f)$. We collect these observations in the following theorem.

Theorem 13.1.13 *If $\xi \in \mathbb{C}$ is an attracting periodic point of a meromorphic function $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$, then $A^*(\xi)$ and $A(\xi)$ are open sets,*

$$\xi \in A^*(\xi) \subset A(\xi) = \bigcup_{n=0}^{\infty} f^{-n}(A^*(\xi)) \subset F(f),$$

and $A^*(\xi)$ is a connected component of the Fatou set $F(f)$.

In particular, the attracting periodic point of a meromorphic function f belongs to its Fatou set. For repelling periodic points, just the opposite is true.

Theorem 13.1.14 *Each repelling periodic point of a meromorphic function $f: \mathbb{C} \to \widehat{\mathbb{C}}$ belongs to the Julia set $J(f)$ of f.*

Proof Seeking contradiction, suppose that a repelling periodic point ξ of f belongs to the Fatou set $F(f)$. Denote the minimal period of ξ by p. Let U be such an open neighborhood of ξ that the family $\left(f^n|_U\right)_{n=0}^{\infty}$ is normal. Then there exists a meromorphic function $g: U \to \widehat{\mathbb{C}}$ to which some sequence $\left(f^{pk_n}|_U\right)_{n=0}^{\infty}$ converges uniformly on compact subsets of U. Then $g(\xi) = \xi$; therefore, g is holomorphic and, in particular, $g'(\xi) \in \mathbb{C}$. But, on the other hand,

$$|g'(\xi)| = \lim_{k\to\infty} \left|\left(f^{pk_n}\right)'(\xi)\right| = \lim_{k\to\infty} |(f^p)'(\xi)|^{k_n} = +\infty$$

as $|(f^p)'(\xi)| > 1$. This contradiction finishes the proof. ■

We also have the following.

Theorem 13.1.15 *Each rationally indifferent periodic point of a meromorphic function $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ belongs to the Julia set $J(f)$ of f.*

Proof Changing coordinates by a translation, we may assume without loss of generality that this periodic point is equal to 0. Passing to a sufficiently high iterate, we may further assume that 0 is a simple parabolic fixed point of f. Then the Taylor series expansion of f about 0 takes on the form

$$f(z) = z + az^{p+1} + \text{ higher terms of } z,$$

where $a \neq 0$ and $p \geq 1$ is an integer. We shall show by induction that

$$f^n(z) = z + naz^{p+1} + \text{ higher terms of } z.$$

Indeed, this is, of course, true for $n = 1$. Assuming that it is true for some $n \geq 1$, and denoting higher than $k \geq 0$ terms of w (a power series of w starting with w^{k+1}) by $HT_k(w)$, we get that

$$\begin{aligned} f^{n+1}(z) = f(f^n(z)) &= f^n(z) + a(f^n(z))^{p+1} + HT_{p+1}(f^n(z)) \\ &= f^n(z) + a(f^n(z))^{p+1} + HT_{p+1}(z) \\ &= z + naz^{p+1} + HT_{p+1}(z) \\ &\quad + a\big(z + naz^{p+1} + HT_{p+1}(z)\big)^{p+1} + HT_{p+1}(z) \\ &= z + (n+1)az^{p+1} + HT_{p+1}(z). \end{aligned}$$

The inductive proof is complete. Consequently,

$$(f^n)^{(p+1)}(0) = a(p+1)!\,n.$$

Therefore, $\lim_{n\to\infty} |(f^n)^{(p+1)}(0)| = +\infty$; as $f(0) = 0$, the proof can now be concluded in the same way as the proof of Theorem 13.1.14. ■

Definition 13.1.16 Let ξ be a periodic point of a meromorphic function $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ with minimal period $p \geq 1$. The map f^p is called *linearizable* near the periodic point ξ if and only if f^p is topologically conjugate to its differential

$$z \longmapsto g(z) := \xi + (f^p)'(\xi)(z - \xi)$$

in some (sufficiently small) neighborhood of ξ.

Theorem 13.1.17 *An irrationally neutral periodic point ξ of a meromorphic function $f: \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ with minimal period $p \geq 1$ belongs to the Fatou set $F(f)$ if and only if f^p is linearizable near ξ. If this holds, the point ξ is called a* Siegel periodic point *of f. The corresponding topological conjugacy then also yields a holomorphic one.*

Proof Replacing f by f^p we may assume without loss of generality that $p = 1$, i.e., that ξ is a fixed point of f. Furthermore, changing coordinates by a translation, we may assume without loss of generality that $\xi = 0$. Write

$$f'(0) := \gamma, \ \ |\gamma| = 1.$$

First, we assume that f is linearizable near 0. This means that

$$H \circ f \circ H^{-1}(z) = \gamma z, \quad z \in D,$$

where D is a sufficiently small disk centered at 0 and $H\colon D \to D$ is a homeomorphism. Iterating this equation, we get, for every $n \geq 0$, that

$$H \circ f^n \circ H^{-1}(z) = \gamma^n z, \quad z \in D.$$

Equivalently,

$$f^n(z) = H^{-1}(\gamma^n H(z)), \quad z \in D.$$

In particular, $f^n(D) \subseteq D$ for every $n \geq 0$. Therefore, the family of iterates $\big(f^n|_D\big)_{n=0}^{\infty}$ is normal, so $0 \in F(f)$.

We now assume that $0 \in F(f)$. Then there is a neighborhood of 0 on which the sequence $(f^n)_{n=0}^{\infty}$ is equicontinuous; from this, we see that there exists some ball $B(0,r)$ $(0 < r < 1)$ of 0 such that, for all $n \geq 0$ and all $z \in U$, we have that

$$|f^n(z)| = |f^n(z) - f^n(0)| < 1. \tag{13.7}$$

Now, for every $n \geq 1$, define a function $T_n\colon B(0,r) \to \mathbb{C}$ by the formula

$$T_n(z) := n^{-1}\big(z + \gamma^{-1} f(z) + \gamma^{-2} f^2(z) \cdots + \gamma^{-(n-1)} f^{n-1}(z)\big).$$

Note that, as $|\gamma| = 1$, we have that

$$T_n(B(0,r)) \subseteq B(0,1) \tag{13.8}$$

for every $n \geq 1$. A direct verification shows that the functions T_n satisfy the following:

$$(n/\gamma)T_n(f(z)) + z = (n+1)T_{n+1}(z) = nT_n(z) + \gamma^{-n} f^n(z).$$

Hence,

$$T_n(f(z)) - \gamma T_n(z) = n^{-1}(\gamma^{1-n} f^n(z) - \gamma z).$$

Since $|\gamma| = 1$ and invoking (13.7), we, thus, conclude that

$$T_n(f(z)) - \gamma T_n(z) \to 0 \tag{13.9}$$

uniformly on $B(0,r)$ as $n \to \infty$. Next, (13.8) implies that the sequence $\big(T_n|_{B(0,r)}\big)_{n=1}^{\infty}$ is normal on U. It, thus, follows that there exists $(k_n)_{n=1}^{\infty}$, an increasing sequence of positive integers, such that $T_{k_n}\colon B(0,r) \longrightarrow B(0,1)$ converges locally uniformly on $B(0,r)$ to some holomorphic function $H\colon B(0,r) \longrightarrow B(0,1)$. By (13.9), it satisfies

$$H \circ f(z) = \gamma H(z)$$

for all $z \in U$. Since $T_n'(0) = 1$, we also have that $H'(0) = 1$, so H is a homeomorphism on a sufficiently small neighborhood of 0. This completes the proof. ■

Definition 13.1.18 An irrationally neutral periodic point ξ of a meromorphic function $f\colon \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ with minimal period $p \geq 1$ belonging to $J(f)$ is called a *Cremer* periodic point of f. If $p \geq 1$ denotes the prime period of ξ, then near ξ the map f^p is not topologically conjugate to its differential (see Theorem 13.1.17).

From all the above, we have the following.

Theorem 13.1.19 *All attracting and Siegel periodic points of a meromorphic function $f\colon \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ are in the Fatou set of f, while repelling, rationally indifferent, and Cremer periodic points are in the Julia set of f.*

Also, a point $\xi \in \widehat{\mathbb{C}}$ is called preperiodic if $f^n(\xi)$ is periodic for some $n \geq 0$.

If f is a nonnearly entire meromorphic function and $n \geq 2$, then f has infinitely many periodic points of mininimal period n. In fact, f has infinitely many repelling periodic points of minimal period n and the Julia set of f is the closure of the set of all repelling periodic points of f. We shall now prove these results. Moreover, the Julia set of f will turn out to be perfect.

Theorem 13.1.20 *If $f\colon \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ is a nonnearly entire meromorphic function, then $J(f)$, the Julia sets of f, is the closure of the set of all repelling periodic points of f.*

In order to prove Theorem 13.1.20 and the preceding statement about repelling periodic points, we need the following well-known "Five-Island Theorem " of Ahlfors (see [**Ah**]) from complex analysis.

Theorem 13.1.21 (Ahlfors Five-Island Theorem) *If $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ is a transcendental meromorphic function and $D_1, D_2, \dots, D_5$ are any five simply connected domains in $\mathbb{C}$ with mutually disjoint closures, then there exists at least one $j \in \{1, 2, \dots, 5\}$ such that, for every $R > 0$, there exists a simply connected domain $G \subseteq \{z \in \mathbb{C}\colon |z| > R\}$ for which the map $f|_G\colon G \to D_j$ is a conformal homeomorphism. If f has only finitely many poles, then the number "5" may be replaced by "3."*

The next lemma follows from Theorem 13.1.21.

Lemma 13.1.22 *Suppose that $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ is a transcendental meromorphic function and that some five points $z_1, z_2, \dots, z_5 \in O^{-1}(\infty)\backslash\{\infty\}$ are mutually*