

PROBLEM SOLVING

Intelligent mental representations of physical, cognitive, and social environments allow humans to navigate enormous search spaces, whose sizes vastly exceed the number of neurons in the human brain. This allows us to solve a wide range of problems, such as the Traveling Salesperson Problem, insight problems, as well as mathematics and physics problems. As an area of research, problem solving has steadily grown over time. Researchers in artificial intelligence have been formulating theories of problem solving for the last seventy years. Psychologists, on the other hand, have focused their efforts on documenting the observed behavior of subjects solving problems. This book represents the first effort to merge the behavioral results of human subjects with formal models of the causative cognitive mechanisms. The first coursebook to deal exclusively with the topic, it provides a main text for elective courses and a supplementary text for courses such as cognitive psychology and neuroscience.

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Cognitive Mechanisms and Formal Models

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I dedicate this book to my wife Irmina Agnieszka

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Preface

This book is intended for undergraduate and graduate courses on *Human Problem Solving*. Note that such a course has rarely been offered at American universities despite the fact that it is as important as such traditional and widely offered courses as *Sensation and Perception* and *Cognitive Psychology*. This book covers insight problem solving, the role of symmetry and invariance in scientific discovery, combinatorial optimization problems, and the contribution of gestalt psychology, especially its emphasis on mental representations. In fact, the mental representation of problems turns out to be *the* underlying theme of the entire book. The first chapter explains why mental representations are necessary in problem solving and the rest of the book describes a wide range of possible representations and their use across all, or almost all, types of problems. The book also includes perceptual and cognitive inferences, which are treated as solutions of constrained optimization problems, the Theory of Mind, mathematics problems, as well as intuitive physics and causal reasoning.

This textbook emphasizes understanding the mathematical and computational mechanisms underlying problem solving. I want the students to learn *what is computed and how it is computed* when problems are solved. The topics, listed above, allow me to explain the theoretical concepts inherent in solving problems. My preferred emphasis on *theory* proves to be fruitful because it provides a relatively coherent, intelligible story. The book also describes many empirical studies, but they only play a supportive role. Structuring the class this way will prepare cognitive psychology students to explore the related area called artificial intelligence, and it will make it easy for computer science and engineering students to venture into the science of the mind.

Keeping this book to a manageable length required me to leave out some material, such as the neuroscience of problem solving, reinforcement learning and decision making. These as well as other topics can easily be added by Instructors when they use this textbook in their classes on problem solving.

The book provides problems to solve and projects to do after each chapter. Some problems are easy, while others are difficult or even very difficult. Each instructor may decide which problems to use. The text throughout the entire book provides many other problems that are solved partially or completely, as well as references to other sources that have additional problems. The book is accompanied by a software library, written in Python, and hosted on *GitHub* at the following link: <https://github.com/jackvandrunen/tsp>. This software was developed by Jacob VanDrunen and it provides tools for solving the Traveling Salesman Problem. The reader can find instructions in this link on how to download and install the library by using Python's package manager, as well as links to

documentation and examples of programs. The library is open source, and technically inclined readers, who find the code useful for their work, are also welcome to “open issues” and “pull requests.”

I would like to thank Robert M. Steinman, who read the entire book and provided me with many suggestions. I would also like to acknowledge the late William H. Batchelder, who insisted that I develop a course at UC Irvine on human problem solving. Finally, I would like to thank Stephen Acerra, the editor at the Cambridge University Press who encouraged me to write and publish this book.