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Relevant Pre-requisites and Terminologies

Before embarking on the formal study of partial differential equations (PDEs), it is essential to equip ourselves with background materials from multivariable calculus, geometry of curves and surfaces, and ordinary differential equations (ODEs; including total and simultaneous total differential equations). Selected ideas from earlier mentioned courses play an important role in the study of PDEs. For the sake of completeness and to make our text self-contained, we briefly discuss the preliminaries as indicated earlier, which are needed in the subsequent chapters. The approach adopted in this chapter is somewhat different from the one used in the rest of the chapters. This chapter is descriptive in nature, wherein the utilised arguments are intended towards plausibility and understanding rather than adopting traditional rigorous ways. The preliminaries are divided into five sections.

1.1 Partial Derivatives and Allied Topics

PDEs involve at least two independent variables. Consequently, the tools of the calculus of several variables are instrumentals in PDEs. In the following lines, we recall some relevant notions and terminologies from differential calculus of two and three variables.

Vectors: An element of Euclidean space $\mathbb{R}^n$ is an n-tuple of the form $\mathbf{x} = (x_1, x_2, \dots, x_n)$, which is called an n-vector or simply, a vector. The real numbers $x_1, x_2, \dots, x_n$ are called components of this element. The sum and scalar multiplication in $\mathbb{R}^n$ are defined as component-wise sum and component-wise scalar multiplication. The dot product of two elements $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ is defined as

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

The *norm* or *length* of a vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ is defined as

$$\left\| \mathbf{x} \right\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

Thus, $\mathbf{x} \cdot \mathbf{x} = \left\| \mathbf{x} \right\|^2$. A vector of length 1 is called a *unit vector*.

Standard Basis Vectors: The two vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are called *standard basis vectors of* $\mathbb{R}^2$. Any vector $u = (a, b) \in \mathbb{R}^2$ can be written as a linear combination of e_1 and e_2, so that

$$u = ae_1 + be_2.$$

Similarly, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$ are called *standard basis vectors of* $\mathbb{R}^3$.

Neighbourhoods: The δ-neighbourhood of a point $(x_0, y_0) \in \mathbb{R}^2$ is an open sphere centred at (x_0, y_0) with radius $\delta > 0$, which can be represented by

$$N_\delta(x_0, y_0) = \{(x, y) \in \mathbb{R}^2 : \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\}.$$

Regions: Let R be a region (subset) of $\mathbb{R}^2$ and (x_0, y_0) a point in R. We say that (x_0, y_0) is an *interior point* of R if there exists a δ-neighbourhood of (x_0, y_0), which lies entirely in R. If every point in R is an interior point, then R is called an *open region*. We say that (x_0, y_0) is a *boundary point* of R if every δ-neighbourhood of (x_0, y_0) contains points that lie outside of R as well as points that lie in R. If R contains all its boundary points, then R is called a *closed region*. The set of all interior points of R is called the *interior* of R and usually it is denoted by R°. Similarly, the set of all boundary points of R is called the *boundary* of R and often it is denoted by ∂R. The *closure* of R, often denoted by $\overline{R}$, remains the union of all boundary points of R with R, that is, $\overline{R} = R \cup \partial R$.

Partial Derivatives: Let $z = f(x, y)$ be a real-valued function of two variables. The partial derivative of z w.r.t. x, denoted by $\dfrac{\partial z}{\partial x}$ or $\dfrac{\partial f}{\partial x}$ or f_x, is defined as

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

provided the limit exists. Similarly, the partial derivative of z w.r.t. y, denoted by $\dfrac{\partial z}{\partial y}$ or $\dfrac{\partial f}{\partial y}$ or f_y, is defined as

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y + k) - f(x, y)}{k}$$

provided the limit exists. The partial derivatives at a particular point (x_0, y_0) are often denoted by

$$\frac{\partial f}{\partial x}(x_0, y_0) \quad \text{or} \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} \quad \text{or} \quad f_x(x_0, y_0)$$

and

$$\frac{\partial f}{\partial y}(x_0, y_0) \quad \text{or} \quad \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} \quad \text{or} \quad f_y(x_0, y_0).$$

Thus, we have

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

and

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{k\to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}.$$

In practice, f_x can be computed as ordinary derivative of f w.r.t. x by treating y as a constant. Similarly, f_y is determined as an ordinary derivative of f w.r.t. y by treating x as a constant.

Notice that f_x and f_y are called *first-order partial derivatives*. In the similar manner, we can define partial derivatives for a function of more than two variables. Thus far, a function $u = f(x, y, z)$ possesses three first-order partial derivatives, namely, u_x, u_y, and u_z.

Higher Order Derivatives: As the partial derivatives are themselves functions, we can take their partial derivatives to obtain higher order derivatives. Higher order derivatives are denoted by the order in which the differentiation occurs. For instance, the function $z = f(x, y)$ admits the following four different second-order partial derivatives.

(i) Differentiate twice w.r.t. x:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right).$$

(ii) Differentiate twice w.r.t. y:

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right).$$

(iii) Differentiate first w.r.t. x and then w.r.t. y:

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}.$$

(iv) Differentiate first w.r.t. y and then w.r.t. x:

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}.$$

The last two derivatives f_{xy} and f_{yx} are called *mixed partial derivatives*. Notice that the mixed partial derivatives are not necessarily equal.

Homogeneous Functions: A function $z = f(x, y)$ is said to be a *homogenous function* of degree n if for every positive real number λ, it satisfies

$$f(\lambda x, \lambda y) = \lambda^n f(x, y).$$

Euler's Theorem: Let $f(x, y)$ be a continuously differentiable and homogenous function of degree n. Then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$$

Moreover, if $f(x, y)$ has continuous second-order partial derivatives, then

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f.$$

Directional Derivatives: Let $z = f(x, y)$ be a function and $u = (\alpha, \beta)$ a unit vector. Then, the *directional derivative* of f at a point (x_0, y_0) in the direction of u, denoted by $D_u f(x_0, y_0)$, is defined as

$$D_u f(x_0, y_0) = \lim_{t\to 0}\frac{f(x_0 + t\alpha, y_0 + t\beta) - f(x_0, y_0)}{t}$$

provided the limit exists. Here, u is called the *direction vector*.

Clearly, $D_{e_1} f(x_0, y_0) = f_x(x_0, y_0)$ and $D_{e_2} f(x_0, y_0) = f_y(x_0, y_0)$, that is, the directional derivatives in the directions of standard basis vectors coincide with partial derivatives. In other words, the partial derivatives are directional derivatives along coordinate axes.

Similarly, we can define the directional derivatives of $u = f(x, y, z)$ at a point (x_0, y_0, z_0) in the direction of unit vector $u = (\alpha, \beta, \gamma)$ as

$$D_u f(x_0, y_0, z_0) = \lim_{t\to 0}\frac{f(x_0 + t\alpha, y_0 + t\beta, z_0 + t\gamma) - f(x_0, y_0, z_0)}{t}.$$

Vector Fields: A *vector field in plane* is a function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$. Thus, we write

$$\mathbf{F}(x, y) = (f_1(x, y), f_2(x, y))$$

wherein f_1 and f_2 are real-valued functions of two variables and are called the components of $\mathbf{F}$.

Similarly, a *vector field in space* is a function $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ so that

$$\mathbf{F}(x, y, z) = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)).$$

Gradient: Let $z = f(x, y)$ be a function such that f_x and f_y exist. Then the *gradient* of f, denoted $\nabla f(x, y)$, is a vector field defined as

$$\nabla f(x, y) = (f_x(x, y), f_y(x, y)).$$

Thus far, the gradient of f at a particular point (x_0, y_0) is

$$\nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0)).$$

Similarly, the gradient of $u = f(x, y, z)$ is defined as

$$\nabla f(x, y, z) = (f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)).$$

Fact: The directional derivative is the dot product of the gradient and direction vector, that is,

$$D_u f = \nabla f \cdot u.$$

Total Differentials: Let $z = f(x, y)$ be a function such that f_x and f_y exist. If dx and dy define the differentials of independent variables, then the *total differential* dz is defined as

$$dz = f_x(x, y)dx + f_y(x, y)dy.$$

Chain Rules: Let $z = f(x, y)$ be a differentiable function of two variables. If x and y are differentiable functions of one independent variable 't', that is, $x = \phi(t)$ and $y = \psi(t)$, then $z = f(\phi(t), \psi(t))$ is a (composite) function of t. Thus, the chain rule states that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

We now consider the situation where the function $z = f(x, y)$ is differentiable but each of x and y is a differentiable function of two variables 's' and 't', that is, $x = \phi(s, t)$ and $y = \psi(s, t)$. Then, z is indirectly a function of s and t and hence the chain rule states that

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

Jacobian: Let $u = u(x, y)$ and $v = v(x, y)$ be two differentiable functions of two independent variables x and y. The *Jacobian* of u and v w.r.t. x and y is the second-order functional determinant defined by

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.$$

Functional Dependence: Two functions $u = u(x, y)$ and $v = v(x, y)$ defined on a set $D \subset \mathbb{R}^2$ are said to be *functionally dependent* or simply, *dependent* if there exists a real-valued continuously differentiable function $F(u, v)$ such that for all $(x, y) \in D$, we have

$$F(u(x, y), v(x, y)) = 0 \qquad \text{and} \qquad \nabla F(u(x, y), v(x, y)) \neq 0.$$

Two functions are called *functionally independent* or *independent* if they are not dependent.

Fact: Two functions $u = u(x, y)$ and $v = v(x, y)$ are independent iff

$$\frac{\partial(u, v)}{\partial(x, y)} \neq 0.$$

Implicit Functions: Let x and y be two variables, which are related by a functional equation of the form $F(x, y) = 0$. A function $y = f(x)$ is called *implicit function* defined by $F(x, y) = 0$ if $F(x, f(x)) = 0$ is satisfied for all x in the domain of f. The equation $F(x, y) = 0$ is called implicit form of $y = f(x)$. The well-known *Implicit Function Theorem* provides the conditions under which $F(x, y) = 0$ defines y implicitly as a function of x. The derivative of $y = f(x)$ can be determined as follows:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

whenever $F_y \neq 0$.

Inverse Function Theorem: Let $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function, which transforms the variables (x, y) to the variables (u, v) so that $u = u(x, y)$ and $v = v(x, y)$. If (x_0, y_0) is a point in the domain of $\mathbf{T}$ such that

$$J\,\mathbf{T}(x_0, y_0) := \left.\frac{\partial(u, v)}{\partial(x, y)}\right|_{(x_0, y_0)} \neq 0$$

and $(u_0, v_0) = \mathbf{T}(x_0, y_0)$, then there exist neighbourhoods N_0 of (x_0, y_0) and N_0' of (u_0, v_0) such that the restriction $\mathbf{T}|_{N_0}$ has an inverse $\mathbf{T}^{-1} : N_0' \to N_0$, which is differentiable at (u_0, v_0) and $J\,\mathbf{T}^{-1}(u_0, v_0) = [J\,\mathbf{T}(x_0, y_0)]^{-1}$.

Invertible Transformation: In lieu of inverse function theorem, a coordinate transformation for which the Jacobian $J \neq 0$ is called an *invertible transformation* or *non-singular transformation*.

1.2 Curves and Surfaces

The integrals of PDEs are surfaces, while the particular solution of an initial value problem remains a surface passing through a given curve. For the appreciation of the methods of solutions and for the interpretation of the solutions of PDEs, a study of curves and surfaces is essential. Henceforth, in the following lines, we briefly discuss the relevant aspects of theory of curves in $\mathbb{R}^2$ and $\mathbb{R}^3$ and surfaces in $\mathbb{R}^3$.

Direction Ratios: If a line in space makes the angles α, β, γ with the axes of x, y, z, respectively, then the quantities $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$ are called the *direction cosines* of the line. Further, the numbers a, b, c proportional to directions cosines, so that $a = \lambda l$, $b = \lambda m$, $c = \lambda n$ for any arbitrary real number λ, are called the *direction ratios* of the line.

Plane Curves: A functional equation involving two variables x and y of the form

$$F(x, y) = 0$$

represents a *plane curve*. The explicit representation of the plane curve is defined by a function of single variable of the form

$$y = f(x).$$

Straight line, circle, ellipse, parabola, and hyperbola are natural examples of plane curves.

Surfaces: A functional equation involving three variables x, y, and z of the form

$$F(x, y, z) = 0 \tag{1.1}$$

represents a *surface*. The explicit representation of the surface is defined by a function of two variables of the form

$$z = f(x, y).$$

Plane: A surface is called a *plane* if it can be represented by the general equation of first degree in x, y, z, that is, the equation of the form

$$ax + by + cz + d = 0. \tag{1.2}$$

Every plane determines a (unique) perpendicular line through the origin to the plane, which is called the *normal* of the plane. It can be pointed out that the coefficients a, b, c in Eq. (1.2) represent the direction ratios of the normal $\mathbf{n}$ of the plane.

If a plane represented by (1.2) passes through a point $P_0(x_0, y_0, z_0)$, then $d = -ax_0 - by_0 - cz_0$ so that (1.2) reduces to

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

which is the equation of the plane passing through P_0 and with the normal having a, b, c as direction ratios.

Conicoid: A surface is called *conicoid* or *quadric* if it can be represented by the general equation of second degree in x, y, z, that is, the equation of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0.$$

Conical Surface: A surface $F(x, y, z) = 0$ is said to be *cone* with vertex $A(x_0, y_0, z_0)$ if for each point $P(x, y, z)$ of the surface and for all $\lambda \in \mathbb{R}$, the point $\lambda P + (1 - \lambda)A$ lies on the surface, that is,

$$F\left(\lambda x + (1 - \lambda)x_0, \lambda y + (1 - \lambda)y_0, \lambda z + (1 - \lambda)z_0\right) = 0.$$

Tangent Plane and Normal to a Surface: Equation of tangent plane to the surface (1.1) at a point $P_0(x_0, y_0, z_0)$ is

$$\left.\frac{\partial F}{\partial x}\right|_{P_0}(x-x_0)+\left.\frac{\partial F}{\partial y}\right|_{P_0}(y-y_0)+\left.\frac{\partial F}{\partial z}\right|_{P_0}(z-z_0)=0.$$

It follows that the components of the gradient $\nabla F = (F_x, F_y, F_y)$ evaluated at a point represent the direction ratios of the normal of the tangent plane and, hence, to the surface at that point. Thus, the equation of the normal to the surface (1.1) at $P_0(x_0, y_0, z_0)$ is

$$\frac{x-x_0}{\left.\frac{\partial F}{\partial x}\right|_{P_0}}=\frac{y-y_0}{\left.\frac{\partial F}{\partial y}\right|_{P_0}}=\frac{z-z_0}{\left.\frac{\partial F}{\partial z}\right|_{P_0}}.$$

Let us consider the equation of a surface in explicit form:

$$z = f(x, y).$$

Denote

$$p := \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} \quad \text{and} \quad q := \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}.$$

If we write $F(x, y, z) \equiv f(x, y) - z = 0$, then we have

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} = p$$
$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} = q$$
$$\frac{\partial F}{\partial z} = -1$$

which implies that at any point (x, y, z), the normal to the surface $z = f(x, y)$ has direction ratios $p, q, -1$ and the equation of tangent plane to this surface is

$$p(x-x_0) + q(y-y_0) = (z-z_0).$$

Family of Surfaces: A family of surfaces usually means an infinite set of surfaces. In most of the cases, the surfaces are all of the same type, for example, all spheres differ only in size or position. If each member of a family of surfaces is attached with an arbitrary constant a, we may represent the whole family by the single equation

$$F(x, y, z, a) = 0. \tag{1.3}$$

Substituting a particular value of a in (1.3), the equation represents a specific surface that is assigned with this value of a. Equation (1.3) is called *one-parameter family of surfaces* and a is called *parameter* of this family. In the similar manner, we can define *two-parameter family of surfaces*, which depends on two arbitrary constants (parameters) a and b, as follows:

$$F(x, y, z, a, b) = 0. \tag{1.4}$$

Envelopes: A surface is called the *envelope* of a family of surfaces if it is touched by all members of the family. The equation of envelope of the one-parameter family (1.3) (if exists) is obtained by eliminating 'a' between the following two equations:

$$\left.\begin{aligned} F &= 0 \\ \frac{\partial F}{\partial a} &= 0. \end{aligned}\right\}$$

Similarly, the equation of envelope of two-parameter family (1.4), if exists, is obtained by eliminating 'a' and 'b' between the following three equations:

$$\left.\begin{aligned} F &= 0 \\ \frac{\partial F}{\partial a} &= 0 \\ \frac{\partial F}{\partial b} &= 0. \end{aligned}\right\}$$

In a manner similar to the family of surfaces, we can define family of curves and their envelope.

Space Curves: A system of two functional equations involving three variables x, y, and z of the form

$$\left.\begin{aligned} F(x, y, z) &= 0 \\ G(x, y, z) &= 0 \end{aligned}\right\}$$

represents a *space curve*. Thus, a space curve can be determined as an intersection of two surfaces.

The following result is used to determine the equation of the surface, which passes through a given space curve.

Theorem 1.1. *Let* Γ *be a space curve represented by*

$$\left.\begin{aligned} u(x, y, z) &= c_1 \\ v(x, y, z) &= c_2. \end{aligned}\right\}$$

Then the functional relation

$$F(u, v) = 0$$

such that $F(c_1, c_2) = 0$ *represents a surface passing through* Γ.

Parameterised Curves: A plane curve can be viewed as a mapping $\gamma : I \subset \mathbb{R} \to \mathbb{R}^2$ so that

$$\gamma(t) = (x(t), y(t))$$

where I is an interval and the components $x(t)$ and $y(t)$ of γ are continuous functions of t. The domain I is called the parameter interval and the variable $t \in I$ is called the parameter of the curve. Hence, the equations

$$x = x(t), y = y(t)$$

are called the parametric equations of the curve.

Similarly, a parameterised space curve can be represented by the function $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ so that

$$\gamma(t) = (x(t), y(t), z(t))$$

whereas the equations

$$x = x(t), y = y(t), z = z(t)$$

are called the parametric equations of the curve. Here, it can be highlighted that any curve can be represented by different sets of parametric equations. Now, we present examples of several well-known curves in their Cartesian as well as parametric forms.

Example 1.1. *Parametric equations of the* **circle** $x^2 + y^2 = a^2$ *are*

$$x = a\cos t,\ y = a\sin t, \quad 0 \le t \le 2\pi.$$

Another possible parametric equations of the circle are

$$x = a\sin t,\ y = a\cos t, \quad \frac{\pi}{2} \le t \le \frac{5\pi}{2}.$$

.

Example 1.2. *Parametric equations of the* **ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *are*

$$x = a\cos t,\ y = b\sin t, \quad 0 \le t \le 2\pi$$

or

$$x = a\sin t,\ y = b\cos t, \quad \frac{\pi}{2} \le t \le \frac{5\pi}{2}.$$

.

Example 1.3. *Parametric equations of the* **hyperbola** $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *are*

$$x = a\sec t,\ y = b\tan t, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$