Four Open Questions for the *N*-Body Problem

The *N*-body problem has been investigated since it was posed by Isaac Newton in 1687. However, vast tracts of the problem remain open. Showcasing the vibrancy of the problem, this book describes four open questions and explores the progress made over the last 20 years. After a comprehensive introduction, each chapter focuses on a different open question, highlighting how the stance taken and tools used vary greatly depending on the question.

Progress on question one, “Are the central configurations finite?”, uses tools from algebraic geometry. Question two, “Are there any stable periodic orbits?”, is dynamical and requires some understanding of the KAM theorem. The third question, “Is every braid realised?”, requires topology and variational methods. The final question, “Does a scattered beam have a dense image?”, is quite new, and formulating it precisely takes some effort.

This book is an excellent resource for students and researchers of mathematics, astronomy, and physics interested in exploring state-of-the-art techniques and perspectives on this classical problem.

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Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment,
a department of the University of Cambridge.

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www.cambridge.org
Information on this title: www.cambridge.org/9781009200585

DOI: 10.1017/9781009200608

When citing this work, please include a reference to the DOI 10.1017/9781009200608

First published 2025

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging-in-Publication Data
Names: Montgomery, Richard, 1956– author.
Title: Four open questions for the n-body problem / Richard Montgomery, University of
California, Santa Cruz.
Description: Cambridge, United Kingdom ; New York, NY : Cambridge University Press,
2025. | Includes bibliographical references and index.
Identifiers: LCCN 2024018152 (print) | LCCN 2024018153 (ebook) |
ISBN 9781009200585 (hardback) | ISBN 9781009200608 (ebook)
Subjects: LCSH: Many-body problem.
Classification: LCC QB362.M3 M66 2025 (print) | LCC QB362.M3 (ebook) |
DDC 521–dc23/eng20240802
LC record available at https://lccn.loc.gov/2024018152
LC ebook record available at https://lccn.loc.gov/2024018153

ISBN 978-1-009-20058-5 Hardback

Cambridge University Press & Assessment
978-1-009-20058-5 — Four Open Questions for the N-Body Problem
Richard Montgomery
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I dedicate this book to the memory of Jerry Marsden, who introduced me to geometric mechanics, and to Bill Burke, who listened to my first seminar on the N-body problem, to Chris Golé, who was also there at that seminar and who insisted that I introduce myself to Alain Albouy and Alain Chenciner, and to Albouy and Chenciner and their institution, the IMCCE, for support, friendship, a sense of history, surprising ideas, and their patience with my French. Finally, I have to dedicate it to my family as well and, in particular, my grandson Jude for his unending fascination and joy at being an intermediary in the interactions of elastic spherical objects with gravity and pavement.

Contents

Preface

The N-body problem, despite having been formulated more than 350 years ago, is alive and well. My goal is to convince you of this by exploring four open questions in the field. Each question gets a chapter, and these make up Chapters 1, 2, 3, and 4. These chapters begin by stating their question. They go on to explain why that question is important, to explore partial answers, and to dive into some of the methods used to arrive at these partial answers.

This book is an outgrowth of a colloquium I gave (online) at Vanderbilt in the Fall of 2020 during the midst of the COVID-19 pandemic. I tried to write the book to be accessible by colleagues in any field of mathematics as well as for strong undergraduates. I expect readers to have proficiency in linear algebra, and to understand what a differential equation is. I hope that the more traditional clientele of experts in celestial mechanics, dynamical systems, and mathematical physics will get something out of the book. I do not expect readers to need a working knowledge of introductory physics. For this reason I've included Chapter 0 and Appendix A, which go into some of the physics, posing the N-body problem as a differential equation, and discussing the assumptions underlying the posing of the problem, as well as its symmetries, conservation laws, and other structures implicit to the equations. Preceding Chapter 0 is Chapter -1, which is a pictorial tour of some known solutions.

Three of these four open questions are stated in the succinct collection of open problems due to Albouy, Cabral, and Santos [6]. The question making up Chapter 1 is their problem 9. The question of Chapter 2 is their problem 7. The question of Chapter 3, for $N = 3$, is (essentially) their problem 4. One will find many other open problems in this collection.

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Humanity has been working on the N-body problem since Newton posed it in 1687 [163]. You might think we'd be done with it by now. We're not. The problem remains very much alive, and this book was written to convince my readers of this fact. Substantial results have been achieved in the last decade, but new open questions continue to arise.

Perturbation theory has dominated the analytical work on the problem. One expands solutions in some series – Taylor, Fourier – about known solutions to limiting cases of the problem, successively working out the terms in the series by various ingenious schemes. Taylor series are expanded with respect to small parameters – typically mass or distance ratios – that arise naturally in the problem. The limiting cases are often some version or other of the two-body problem. Laplace, Lagrange, Poisson, Gauss, Legendre, Delanay, Hill, Poincaré, Moser, Arnol'd – the names go on and on – have all contributed to perturbation theory with great success. I have avoided perturbation theory in this book as well as in my own career. I cannot compete with the old masters and would likely end up with nothing new to say. An exception to my "avoid perturbation theory" rule is the discussion of KAM theory in Chapter 2.

My favorite case of the N-body problem is the equal mass zero angular momentum planar three-body problem. Studying it forces one to replace perturbation theory with other tools. All four open questions except the first one are open for this case. The third open question has been solved for the equal mass planar three-body problem when the angular momentum is small but nonzero.

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Newton [163] posited that any two masses in the universe attract each other by a gravitational force. He took this force to be proportional to $1/r^2$ where r is the distance between the masses. Supposing the universe to be populated by a finite number N of point masses subject only to their mutual gravitational attractions and his laws of mechanics, he formulated a set of differential equations (see Equation (0.4) in Chapter 0) that describe the motions of these N point masses. **The classical N-body problem is the study of this dynamical system.**

Newton solved his two-body problem. Supposing these two masses to be the Sun and a planet, he derived Kepler's three laws of planetary motion and thereby gained himself a preeminent role in the history of science. See Section −1.1.

More than two hundred years later, Poincaré proved that a limiting case of the three-body problem is unsolvable in a certain technical sense: it admits "homoclinic tangles" and therefore is not "integrable by analytic functions." The effect of his proving unsolvability was analogous to that of Galois' proving the unsolvability of the general quintic. Rather than killing their subjects, they developed methods to establish their impossibility results that opened up vast and previously unimagined vistas of research. For Galois these vistas included group theory, algebra, and number theory. For Poincaré the vistas were nonlinear qualitative dynamical systems (popularly referred to as "chaos theory") and their interaction with topology and analysis.