Continuous Groups for Physicists

The theory of groups and group representations is an important part of mathematics with applications in other areas of mathematics as well as in physics. It is basic to the study of symmetries of physical systems. Its mathematical concepts are equally significant in understanding complex physical systems. It offers the necessary tools to describe, for instance, crystal structures, elementary particles with spin, both Galilean symmetric and special relativistic quantum mechanics, the fundamental properties of canonical commutation relations and spinor representations of orthogonal groups extensively used in quantum field theory.

Continuous Groups for Physicists introduces the ideas of continuous groups and their applications to graduate students and researchers in theoretical physics. The book begins with an introduction to groups and group representations in the context of finite groups. This is followed by a chapter on the special algebraic features of the symmetric groups. The authors then present the theory of Lie groups, Lie algebras and in particular the classical families of compact simple Lie groups and their representations. Several interesting topics not often found in standard physics texts are then presented: the spinor representations of the real orthogonal groups, the real symplectic groups in even dimensions, induced representations, the Schwinger representation concept, the Wigner theorem on symmetry operations in quantum mechanics, and the Euclidean, Galilei, Lorentz and Poincaré groups associated with spacetime. The general methods and notions of quantum mechanics are used as background throughout.

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In Memoriam
Sophus Lie, Elie Cartan, Hermann Weyl,
Eugene Wigner and Valentine Bargmann – who showed us the way

‘Beauty is truth, truth beauty
That is all ye know on earth,
And all ye need to know’

– John Keats (1795–1821)

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Preface

It has rightly been said that the mathematical theory of groups and group representations is a magnificent gift of nineteenth century mathematics to twentieth century physics. While this is particularly true within the framework of quantum mechanics, with the passage of time its relevance within classical physics has also become well understood and greatly appreciated. Today the importance of group theoretical ideas and methods for physics can hardly be overemphasised; and over the past century or so, a veritable profusion of books devoted to this theme, many of them gems of the literature, have appeared.

The present monograph is primarily based on lectures given by one of us (NM) at the Institute of Mathematical Sciences in Chennai, India, in the Fall of 2007. The lectures were prepared and presented at the invitation of Rajiah Simon, to whom both authors are indebted for his support and encouragement.

The course was titled 'Continuous Groups for Physicists' and consisted of about 45 extended lectures over a two month period. Its aim was to introduce the basic ideas of continuous groups and some of their applications to an audience of post graduate and doctoral students in theoretical physics. After an introduction to the basic ideas of groups and group representations (mainly in the context of finite groups and compact Lie groups), the course presented a selection of useful, interesting and quite sophisticated specific topics not often included in standard courses in physics curricula. The methods and concepts of quantum mechanics served as a backdrop for all the lectures.

The real rotation groups in two and three dimensions are followed by an account of the structures of Lie groups and Lie algebras, and then a description of the compact simple Lie groups. Their irreducible representations are described in some detail. Some of the 'non standard' topics that follow are: spinor representations of real orthogonal groups in both even and odd dimensions; the notion of the 'Schwinger' representation of a group with examples, induced representations, and systems of generalised coherent states; the properties and uses of the real symplectic groups, which are defined only in real even dimensions, and their metaplectic covering group, in a quantum mechanical setting; and the Wigner Theorem on the representation of symmetry operations in quantum mechanics. For the sake of completeness an account of the representation theory of the permutation groups, with its many algebraic features, has been added essentially at the beginning.

While this monograph is not intended to be a text book in the traditional sense, it is hoped that readers will find it useful in that a succinct account of the basics is followed by short accounts of the special topics mentioned above.

References have been listed at the end of each chapter. These include some classics of the literature, a few texts which we have found to be particularly well written, and in some cases a few journal publications. Some of the references appear at the conclusion of more than one chapter. Two types of references have been provided – those useful for the chapter as a whole, and those relevant at specific points in the chapter. Only the latter are indicated in the text, by author name and year of publication.

Some problems are given at the end of each chapter. To help with the more challenging ones, references to books or original papers are included.

Abbreviations

BI	Bargmann Invariant
CCR	Canonical Commutation Relation
CSLA	Compact Simple Lie Algebra
GCS	Generalised Coherent States
Irrep.	Irreducible Representation
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
UIR	Unitary Irreducible Representation
UR	Unitary Representation