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Basic Group Theory and Representation Theory

In this chapter, we present the basic theory of finite groups and their representations as preparation for the discussion of continuous groups that starts from Chapter 3. It is assumed that readers know the basics of set theory, vector spaces, transformations, linear operators, matrix representations, inner products and such. These will be called upon as and when needed.

1.1 Definition of a Group

A group G is a set of elements $a, b, c, \cdots, g, g', \cdots, e, \cdots$ along with a composition (or 'multiplication') law obeying four conditions:

(i) Closure: $a, b \in G \rightarrow ab =$ unique product element $\in G$.

(ii) Associativity: for any $a, b, c \in G$,

$$a(bc) = (ab)c = abc \in G.$$

(iii) Identity: there is a unique element $e \in G$ such that

$$ae = ea = a, \qquad \text{for any } a \in G.$$

(iv) Inverses: for each $a \in G$, there is a unique $a^{-1} \in G$, the inverse of a, such that

$$a^{-1}a = aa^{-1} = e. \tag{1.1}$$

The composition rule or law can be called a binary law as the product is defined for any *pair* of elements. The conditions in (1.1) could be stated in more economical forms, for instance introducing only a left identity and left inverses, and then showing that the more general properties in (1.1) do hold.

One can immediately think of various qualitatively different possibilities. The number of (distinct) elements in G may be finite. Then this number, denoted by $|G|$, is called the order of G. Some other possibilities are that the number of elements may be a discrete infinity, or else a continuous infinity with G being a manifold of some dimension.

1.2 Some Examples

(i) The symmetric group, the group of permutations on n objects, is finite, of order $n!$, and is denoted by S_n. We mention only a few pertinent properties now, and go into some more detail in Chapter 2. Each $p \in S_n$ can be written in several convenient ways:

$$\begin{aligned} p &= \begin{pmatrix} 1 & 2 & \cdots & n \\ p(1) & p(2) & \cdots & p(n) \end{pmatrix} = \begin{pmatrix} j \\ p(j) \end{pmatrix}_{j=1\cdots n} \\ &= (1\; p(1)\; p(p(1)) \cdots)\,(j\; p(j)\; p(p(j)) \cdots) \cdots (k\; p(k)\; p(p(k)) \cdots). \end{aligned} \tag{1.2}$$

In the first form the columns can be rearranged at will, while in the second form the factors and their entries are distinct. The composition law can be developed as follows:

$$qp = \begin{pmatrix} k \\ q(k) \end{pmatrix} \begin{pmatrix} j \\ p(j) \end{pmatrix} = \begin{pmatrix} p(j) \\ q(p(j)) \end{pmatrix} \begin{pmatrix} j \\ p(j) \end{pmatrix} = \begin{pmatrix} j \\ q(p(j)) \end{pmatrix},$$

i.e.,

$$(qp)(j) = q(p(j)), \quad j = 1, 2, \cdots, n. \tag{1.3}$$

We have followed here the rule of 'reading from right to left'. The identity and inverses are:

$$e = \begin{pmatrix} 1 \; 2 \cdots n \\ 1 \; 2 \cdots n \end{pmatrix} = \begin{pmatrix} j \\ j \end{pmatrix}_{j=1\cdots n} = (1)(2)\cdots(n);$$

$$p^{-1} = \begin{pmatrix} p(j) \\ j \end{pmatrix}_{j=1\cdots n}, \quad \text{i.e.,} \quad p^{-1}(p(j)) = j. \tag{1.4}$$

All the conditions in (1.1) can and should be checked.

(ii) All integers with respect to addition. Here the group 'multiplication' law is arithmetic addition. The identity is 0, and inverses are negatives.

(iii) All positive real (or rational) numbers with respect to multiplication. Now the identity is 1, and inverses are reciprocals.

(iv) All vectors in any (real or complex) linear vector space with respect to vector addition. Similar to (ii) above, the identity is the zero vector $\mathbf{0}$, and inverses are negatives.

(v) $SO(2)$ and $O(2)$, $SO(3)$ and $O(3)$: these are the proper and full groups of rotations in a plane or in three dimensions, respectively. These are continuous groups with infinitely many elements. In the $O(2)$ and $O(3)$ cases, the group is made up of two disjoint components, each of which is continuous and connected in an obvious intuitive sense. We study these in Chapter 3.
We can also consider discrete subsets of these, leaving invariant some given regular plane figure or solid. These are the point groups.

(vi) The rotation groups $SO(3), O(3)$ generalise to any dimension n and to the complex case. Thus we arrive at the groups $SO(n)$ and $O(n)$, $SU(n)$ and $U(n)$ for various integers n, which we will study in some detail later. Yet others are the complex orthogonal groups $SO(n, \mathbb{C})$.

(vii) Groups related to spacetime. Here we have the Euclidean, Galilean, Lorentz, and Poincaré groups. All of these are continuous groups with infinitely many elements. They have more than one connected component if discrete operations like space and/or time reflections are included. We will study some of these groups in Chapter 10.

A group G is said to be abelian if for any pair of elements a and b, $ab = ba$. Otherwise it is nonabelian. In the examples listed above, (ii), (iii) and (iv) are abelian.

Among the symmetric groups, S_2 is abelian while S_n for $n \geq 3$ is nonabelian. $SO(2)$ is abelian, and $O(2)$, $SO(3)$ and $O(3)$ are all nonabelian.

The existence and properties of a unique identity element and unique inverse a^{-1} for each $a \in G$ lead to the useful cancellation rules:

$$ab = ac \Leftrightarrow b = c; \quad ba = ca \Leftrightarrow b = c. \tag{1.5}$$

For a finite group G the composition law or entire structure can be displayed in a *multiplication table* with $|G|$ 'rows' and 'columns' labelled by the group elements:

$$\begin{array}{c|cccc} & e & \cdots & b & \cdots \\ \hline e & e & \cdots & b & \cdots \\ \vdots & \vdots & & \vdots & \\ a & a & \cdots & ab & \cdots \\ \vdots & \vdots & & \vdots & \end{array} \tag{1.6}$$

At the intersection of row a and column b stands the product ab. These entries must be consistent with the group laws (1.1). So by (1.5) each row (column) contains every element of G exactly once.

1.3 Operations within a Group

For the present we continue to have in mind a finite group, though many concepts we will go on to introduce are more generally meaningful. For any elements $a, b \in G$, conjugation of a by b leads to another element $a' \in G$:

$$a' = bab^{-1}, \ a = b^{-1}a'b. \tag{1.7}$$

We then say a and a' are conjugate to one another. For each $a \in G$, its *equivalence class or conjugacy class* $\mathcal{C}(a)$ consists of all a' conjugate to a:

$$\mathcal{C}(a) = \text{equivalence class of } a = \{bab^{-1} | \ b \in G, \ \text{omit repetitions}\} \subset G. \tag{1.8}$$

Different classes are generally of different 'sizes'. For instance, $\mathcal{C}(e) = \{e\}$ consists of one element alone. It is easy and important to check:

(i) $\mathcal{C}(a) = \mathcal{C}(bab^{-1})$, any b, so $\mathcal{C}(a)$ is determined by any one of its elements;

(ii) $\mathcal{C}(a) \cap \mathcal{C}(b) = \mathcal{C}(a)$ if $b \in \mathcal{C}(a)$, null otherwise, so two classes cannot overlap partially;

(iii) G is the union of *disjoint* equivalence classes. (1.9)

In S_n, for example, all elements in one class have common cycle structure and vice versa. Thus if we write any $p \in S_n$ in the form

$$p = \underbrace{(i_1\ p(i_1)\ p(p(i_1))\cdots)}_{m_1}\ \underbrace{(i_2\ p(i_2)\cdots)}_{m_2}\cdots,$$

$$n = m_1 + m_2 \cdots, \tag{1.10}$$

where i_2 is not one of the earlier m_1 entries, i_3 is not one of the earlier $m_1 + m_2$ entries, $\cdots$, then $m_1, m_2, \cdots$ is some partition of n. The cycle structure of p is denoted by $(m_1, m_2, \cdots)$, and cycle structures determine equivalence classes and conversely. All $p' \in S_n$ conjugate to p in (1.10) have the same cycle structure as p, and conversely.

In $SO(3)$, as we will recall later, each rotation is by some right handed angle about some axis. Then each class consists of all rotations by a given angle about all possible axes.

Subgroups

A subset $H \subset G$ is a *subgroup* if its elements obey all the conditions to be a group, given the composition law in G. $H = \{e\}$ or G are trivial cases. An elegant and compact criterion for a subset to be a subgroup is this:

$$H \subset G \text{ is a subgroup} \Leftrightarrow \text{for all } h_1, h_2 \in H,\ h_1^{-1}h_2 \in H. \tag{1.11}$$

As examples of subgroups in familiar groups we have: all even integers in the additive group of all integers; all even permutations in S_n making up the alternating subgroup A_n; S_{n-1} within S_n, i.e., permutations p with $p(n) = n$; all rotations in $SO(3)$ about a given axis.

In a finite group, each element leads via its 'powers' to a subgroup called its *cycle*:

$$a \in G \to \{e, a, a^2, \cdots, a^{q-1}\} = \text{cycle of } a = \text{subgroup in } G,$$

$$q = \text{least positive integer such that } a^q = e, q = \text{order of } a. \tag{1.12}$$

We will see that q divides $|G|$; it is clear that $a^{-1} = a^{q-1}$, etc.

Given a subgroup $H \subset G$ and any $g \in G$, conjugation leads to another *conjugate subgroup*:

$$H_g = gHg^{-1} = \{ghg^{-1} |\ h \in H \text{ varying, } g \text{ fixed}\} \subset G. \tag{1.13}$$

Of course, $g \in H$ implies $H_g = H$; and if $g \notin H$, H_g could be different from H.

Cosets

Given a subgroup $H \subset G$, there are two generally distinct and useful ways to break G up into subsets, based on two kinds of cosets:

$$\begin{aligned} aH &= \{ah \mid a \text{ fixed, all } h \in H\} = \text{ right coset containing } a \in G; \\ Ha &= \{ha \mid a \text{ fixed, all } h \in H\} = \text{ left coset containing } a \in G. \end{aligned} \tag{1.14}$$

As we saw with equivalence classes, here too it is easy to see the following: each coset is determined by any one of its elements; two cosets (both right or both left) either coincide fully or are disjoint; G is a union of (right or left) cosets. However, unlike classes, each coset is of the same 'size', namely as 'big' as H. It follows that $|G|/|H|$ is an integer, the number of disjoint right (or left) cosets. This is *Lagrange's theorem*. The claim made after (1.12) is now understood.

For a general subgroup $H \subset G$, right and left cosets are different, leading to different ways of expressing G as a union of disjoint subsets. However, if $H_g = H$ for all $g \in G$, i.e., H is self conjugate, then every right coset is also a left coset and vice versa. Then we say H is a *normal* or an *invariant* subgroup:

$$\begin{aligned} H \text{ is an invariant subgroup } &\Leftrightarrow H_g = gHg^{-1} = H \text{ for every } g \in G \\ &\Leftrightarrow aH = Ha \text{ for every } a \in G \\ &\Leftrightarrow \text{ the two kinds of cosets coincide.} \end{aligned} \tag{1.15}$$

Then these (common) cosets can themselves be regarded as elements of a group, the *quotient group* or *factor group* G/H: the identity is the coset containing the identity element, $eH = H$, ie, H itself; the composition of two cosets is given by $aH \cdot bH = ab \cdot H$; and for inverses we take $(aH)^{-1} = a^{-1}H$. It is easy to check that all the group laws are obeyed. The orders obey $|G/H| = |G|/|H|$.

For any group G, there are two natural invariant subgroups, the *centre* and the *commutator* subgroup. The former consists of all those elements which 'commute' with all elements,

$$Z = \text{centre of } G = \{a \in G \mid ab = ba \text{ for all } b \in G\}, \tag{1.16}$$

so obviously it is abelian. The latter is more intricate and in fact is a way of 'measuring' the extent to which G is nonabelian. If G is abelian, then always $ab = ba$. In general,

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we define the *commutator* of any two elements a and b as the element

$$q(a,b) = aba^{-1}b^{-1} = \text{measure of noncommutativity of } a \text{ and } b,$$
$$q(a,b) = e \Leftrightarrow ab = ba. \tag{1.17}$$

It is easy to see that under inversion and conjugation,

$$q(a,b)^{-1} = q(b,a),\ cq(a,b)c^{-1} = q(cac^{-1}, cbc^{-1}). \tag{1.18}$$

The commutator subgroup $Q \subset G$ is now defined as consisting of products of any numbers of commutator factors:

$$Q = \{q(a_1,b_1)q(a_2,b_2)\cdots q(a_m,b_m)|\ \text{any } m, a\text{'s},\ b\text{'s}\}. \tag{1.19}$$

It is easy to see from (1.18) that Q is an invariant subgroup of G. Further, since $ab = q(a,b)ba = baq(a^{-1}, b^{-1})$, we see that G/Q is abelian. Thus in the quotient all noncommutativity in G has been removed.

There is a converse to this statement: if H is an invariant subgroup in G such that G/H is abelian, then H contains Q.

For the symmetric group S_n, Q is the alternating group A_n of all even permutations, and S_n/A_n is the two element abelian group.

We mention that the concept of the commutator subgroup is basic to the definitions of so-called solvable and nilpotent groups, but we do not go into them here.

Returning to the general concept of invariant subgroups, we have two important definitions:

(a) G is simple $\Leftrightarrow$ G has no (proper) invariant subgroup;

(b) G is semisimple $\Leftrightarrow$ G has no invariant abelian subgroup. (1.20)

This leads to a one-way relationship: G is simple $\Rightarrow$ G is semisimple.

Among the symmetric groups S_n we always have the alternating subgroup A_n which is invariant, so all S_n are nonsimple. For $n = 3$ or $n \geq 5$, A_n is the only invariant subgroup in S_n. In the case of S_4, apart from A_4 there is one other invariant subgroup K of 4 elements, and the quotient $S_4/K \sim S_3$. In contrast, the rotation group $SO(3)$ is simple.

The last operation within a given group we consider is that of an *automorphism*. Given G, an automorphism of G is a map $\tau : G \rightarrow G$ which is one-to-one, onto, invertible and preserves products:

$$a \in G \to \tau(a) \in G: \tau(a)\tau(b) = \tau(ab),\ \tau^{-1} \text{ exists}, \tau(G) = G. \tag{1.21}$$

It is easy to see that $\tau(a)^{-1} = \tau(a^{-1}),\ \tau(e) = e$. Conjugation by a fixed $g \in G$ is an automorphism:

$$g \in G, \text{ fixed: } \tau_g(a) = gag^{-1},\ \ a \in G. \tag{1.22}$$

The conditions in (1.21) are immediately verified. These are called *inner* automorphisms, all others are *outer*. Clearly if G is abelian, every nontrivial automorphism is outer. We will later come across physically important examples of automorphisms.

1.4 Operations with and Relations between Groups

We consider four of these:

(A) *Homomorphism* Given two groups G' and G, a homomorphism is a map $\Phi : G' \to G$ such that images of products are products of images:

$$\Phi(a'b') = \Phi(a')\Phi(b'),\ \text{ all } a', b' \in G'. \tag{1.23}$$

It is easy to see that $\Phi(e') = e \in G$, $\Phi(a'^{-1}) = \Phi(a')^{-1}$, $\Phi(G') \subset G$ is a subgroup in G. We may generally assume $\Phi(G') = G$ or else limit ourselves to $\Phi(G')$ in G. The kernel of a homomorphism is

$$K = \{g' \in G' |\ \Phi(g') = e\} = \text{invariant subgroup in } G', \tag{1.24}$$

so we can form the quotient G'/K.

(B) *Isomorphism* This is a particular case of homomorphism when Φ is one to one, onto and invertible, so $\Phi(G') = G$. Thus as groups G' and G are 'identical' or essentially the same. Returning to a general homomorphism, case (A), we see that, assuming $\Phi(G') = G$,

$$G'/K \text{ is isomorphic to } G. \tag{1.25}$$

In an isomorphism G' and G play symmetric roles, but in a homomorphism this is not so.

(C) *Direct product* Given two groups G_1 and G_2, their direct (or Cartesian) product $G_1 \times G_2$ is another group. Its elements are ordered pairs (a_1, a_2) with $a_1 \in G_1$, $a_2 \in G_2$. The group law is trivial, each entry 'minding its own business':

$$\begin{aligned}(a_1, a_2)(b_1, b_2) &= (a_1 b_1, a_2 b_2);\\ \text{identity} &= (e_1, e_2);\\ (a_1, a_2)^{-1} &= (a_1^{-1}, a_2^{-1}).\end{aligned} \tag{1.26}$$

The orders multiply: $|G_1 \times G_2| = |G_1||G_2|$. In a natural sense, G_1 and G_2 are invariant subgroups in $G_1 \times G_2$ and are recoverable as factor groups with respect to the appropriate kernels.

(D) *Semidirect product* This is a more intricate way of combining G_1 and G_2, a direct product with a 'twist', in which G_1 and G_2 are not treated symmetrically. For each $a_1 \in G_1$, we need an automorphism τ_{a_1} of G_2 obeying certain conditions:

$$\begin{aligned}\tau_{a_1'}(\tau_{a_1}(a_2)) &= \tau_{a_1' a_1}(a_2),\\ \tau_{a_1^{-1}} &= \tau_{a_1}^{-1}.\end{aligned} \tag{1.27}$$

Then the group law for ordered pairs is:

$$G_1 \rtimes G_2 : (a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 \tau_{a_1}(b_2)). \tag{1.28}$$

It is instructive to verify the laws of group structure here; it is not as trivial as with the direct product. In a natural manner we do find that G_1 is a subgroup, G_2 is an invariant one, and the quotient $G_1 \rtimes G_2/G_2 \simeq G_1$.

We will see several physically important examples of semidirect products, especially in Chapter 10.

1.5 Realisations and Representations of Groups

A realisation of a group G arises in the following way. We have a set X, and for each $g \in G$, a map $\phi_g : X \to X$ obeying the group and other laws:

$$\begin{aligned}&\text{(i)} \;\; \text{each } \phi_g \text{ one-to-one, onto, invertible;}\\ &\text{(ii)} \;\; \phi_e = \mathrm{Id}_X, \;\text{identity map;}\\ &\text{(iii)} \;\; \phi_{g'} \circ \phi_g = \phi_{g'g}, \;\text{all } g', g \in G;\\ \text{so}\;\; &\text{(iv)} \;\; \phi_{g^{-1}} = (\phi_g)^{-1}.\end{aligned} \tag{1.29}$$

(For composition of maps, associativity is automatic; in a group in the abstract, it is explicitly postulated). Then we say we have a realisation of G by maps on X.

In the context of realisation of a group G on a set X as defined above, the following notions naturally arise:

$$\text{Orbit of } x \in X : \vartheta(x) = \{x' \in X \mid x' = \phi_g(x), \text{ some } g \in G\}$$

$$\text{Equivalence relation on } X : x \sim y \Leftrightarrow \phi_g(x) = y \text{ for some } g \in G$$

$$\text{Stability (Isotropy) group } H(x) \text{ of } x \in X : H(x) = \{g \in G \mid \phi_g(x) = x\}$$

$$\text{Fixed points } X_g \text{ of } g \in G : X_g = \{x \in X \mid \phi_g(x) = x \text{ for a fixed } g \in G\}$$

In classical Hamiltonian mechanics, groups usually act as canonical transformations on phase space. In quantum mechanics, the state space is a linear vector space, a Hilbert space, and we have the Superposition Principle. In this context, linear representations of groups become relevant. We will hereafter study only these.

1.6 Group Representations

These are particular cases of realisations, with added features. Given a group G and a (real or complex) linear vector space V, we have a representation D of G on V if the following hold:

$$\begin{aligned} &\text{(i)} \quad \text{For each } g \in G,\ D(g) = \text{ invertible linear operator on } V;\\ &\text{(ii)} \quad D(e) = \mathbb{1} = \text{ identity or unit operator on } V;\\ &\text{(iii)} \quad D(g')D(g) = D(g'g), \text{ all } g', g \in G;\\ \text{so } &\text{(iv)} \quad D(g^{-1}) = D(g)^{-1}. \end{aligned} \tag{1.30}$$

The $D(g)$ are also called linear transformations on V. The representation is faithful if

$$g' \neq g \Rightarrow D(g') \neq D(g), \tag{1.31}$$

otherwise it is nonfaithful. The dimension of the representation D is that of V; it may be finite or infinite.

A group G generally has many representations, on various V, of various dimensions.

Given a realisation of a group G on a set X, not naturally endowed with a vector space structure, one can elevate it to a representation of G by defining a vector space $\mathbb{F}[X]$ over a field $\mathbb{F}$ consisting of all formal linear combinations $\sum_i c_i x_i$ of elements