Happy Birthday

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It is our great pleasure and honour to say Happy Birthday to our friend Miles Reid, for your 70th birthday and indeed a few subsequent ones; it takes a long time to make a big pot, as they say, perhaps especially when there are so many potters.

These 11 papers by 20 authors give some idea of the wide range of subject areas, people and countries you have visited and influenced. It would have been easy to fill a book several times over with papers from your friends, and we regret only that we had to stop at some point while there were still so many to ask.

It may be traditional for the introduction of a Festschrift to survey the mathematics where the maestro made their major contributions, but fortunately we are spared that exercise by your own regular surveys [Rei87b, Rei87, Rei00, Rei02a, Rei02b], from the tendencious to the congressional and from the young to the old. There is also your own webpage, which knows nothing of personal data protection and reveals that in the time it has taken to compile this volume your family has grown by several grandchildren.

Nevertheless, to run through a few of the blockbusters, we first learned about canonical singularities from [Rei80], and minimal models became reality in [Rei83a, Rei83b]. Shortly afterwards 'Reid's fantasy' was revealed in [Rei87a]. Several papers on the McKay correspondence include your first coauthored paper [IR96] and another joint paper whose arXiv title 'Mukai implies McKay' is more memorable than the one the journal preferred [BKR01].

Scattered throughout are various long-running obsessions. There are always surfaces, from the smallest invariants [Rei78], to the nonnormal [Rei94], to the positively characteristic [Rei19]. And lurking nearby are codimension 4 Gorenstein rings, projection and its converse unprojection [PR04], and their applications to constructing varieties and maps [CPR00, BKR12, BR13] culminating in the general structure theory [Rei15]. Since the latter paper promises to raise more questions than it answers, perhaps we can guess some of the things you will be doing in the coming decade.

With your eighth decade well under way, it is time to say:

Happy 70th Birthday, Miles!

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On Stable Cohomology of Central Extensions of Elementary Abelian Groups

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We feel honoured to dedicate this article to our friend and colleague Miles Reid on the occasion of his 70th birthday.

Abstract

We consider that kernels of inflation maps associated with extraspecial p-groups in stable group cohomology are generated by their degree-2 components. This turns out to be true if the prime is large enough compared to the rank of the elementary abelian quotient, but false in general.

1 Introduction and Statement of Results

Throughout k will be an algebraically closed field of characteristic $l \geq 0$ and p will be a prime number assumed to be different from l if l is positive. Let G be a finite p-group. One defines the stable cohomology $H^*_{\mathrm{s},k}(G, \mathbb{Z}/p) = H^*_{\mathrm{s}}(G, \mathbb{Z}/p)$ in the following way (this does depend on k, but we suppress it from the notation when there is no risk of confusion): for a finite-dimensional generically free linear G-representation V, let $V^L \subset V$ be the open subset where G acts freely. Then the ideal $I_{G,\mathrm{unstable}}$ in the group cohomology ring $H^*(G, \mathbb{Z}/p)$ is defined to be, equivalently, the kernel of the natural homomorphism:

$$H^*(G, \mathbb{Z}/p) \longrightarrow H^*(\mathrm{Gal}(k(V/G)), \mathbb{Z}/p) \qquad (1.1)$$

or, more geometrically, the kernel of

$$H^*(G, \mathbb{Z}/p) \longrightarrow \varinjlim_{U \subset V^L/G} H^*_{\text{ét}}(U, \mathbb{Z}/p),$$

where U runs over all nonempty Zariski open subsets of V^L/G.

Definition 1.1. We define $H^*_{\mathrm{s}}(G, \mathbb{Z}/p)$ as

$$H^*_{\mathrm{s}}(G, \mathbb{Z}/p) = H^*(G, \mathbb{Z}/p)/I_{G,\mathrm{unstable}}$$

A priori, this seems to depend on the choice of V, but really does not [Bog07, Theorem 6.8]. We often identify $H^*_{\mathrm{s}}(G, \mathbb{Z}/p)$ with its image in $H^*(\mathrm{Gal}(k(V/G)), \mathbb{Z}/p)$.

$H^*_{\mathrm{s}}(G, \mathbb{Z}/p)$ is contravariant in the group G: if $\varphi\colon G' \to G$ is a group homomorphism, V' and V are generically free G' and G-representations with a dominant intertwining map $\Phi\colon V' \to V$ (meaning $\Phi(g'v') = \varphi(g')\Phi(v')$ for all $g' \in G'$, $v' \in V'$), and $U' \subset V'$ and $U \subset V$ are nonempty G', G-invariant open subsets with $\Phi(U') \subset U$, then the diagram

$$\begin{array}{ccc} BG' & \longrightarrow & BG \\ \uparrow & & \uparrow \\ U'/G' & \longrightarrow & U/G \end{array}$$

induces a homomorphism $H^*_{\mathrm{s}}(G, \mathbb{Z}/p) \to H^*_{\mathrm{s}}(G', \mathbb{Z}/p)$ independent of all choices. For a subgroup $H = G'$ of G and φ the inclusion, we call the induced homomorphism $H^*_{\mathrm{s}}(G, \mathbb{Z}/p) \to H^*_{\mathrm{s}}(H, \mathbb{Z}/p)$ restriction or inflation. We also sometimes say that a class in $H^*_{\mathrm{s}}(G', \mathbb{Z}/p)$ is induced from G when it is in the image of $H^*_{\mathrm{s}}(G, \mathbb{Z}/p) \to H^*_{\mathrm{s}}(G', \mathbb{Z}/p)$ and if this map is implied unambiguously by the context.

Definition 1.2. Put $L = k(V/G)$. The unramified group cohomology

$$H^*_{\mathrm{nr}}(G, \mathbb{Z}/p) \subset H^*_{\mathrm{s}}(G, \mathbb{Z}/p)$$

is defined as the intersection, inside $H^*(L, \mathbb{Z}/p)$, of $H^*_{\mathrm{s}}(G, \mathbb{Z}/p)$ and $H^*_{\mathrm{nr}}(L, \mathbb{Z}/p)$; here, as usual, $H^*_{\mathrm{nr}}(L, \mathbb{Z}/p)$ are those classes that are in the kernel of all residue maps associated with divisorial valuations of L, that is, those corresponding to a prime divisor on some normal model of L.

In this article, we study a rather special class of groups.

Definition 1.3. For a prime p, an extraspecial p-group G is a p-group such that its center $Z(G)$ is cyclic of order p and $G/Z(G)$ is a non-trivial elementary abelian group.

This differs a bit from the arguably most common definition using the Frattini subgroup [Suz86, 4., Section 4, Definition 4.14], but it is equivalent to it by [Suz86, 4., Section 4, 4.16].

Thus, each extraspecial p-group sits in an exact sequence

$$1 \longrightarrow Z \longrightarrow G \xrightarrow{\pi} E \longrightarrow 1, \tag{1.2}$$

where $Z \simeq \mathbb{Z}/p$ is the center of the group G and $E \simeq (\mathbb{Z}/p)^n$ is an elementary abelian. Moreover, the skew form given by taking the commutator of lifts of elements in E,

$$\omega\colon E \times E \to Z,$$
$$(x, y) \mapsto [\tilde{x}, \tilde{y}],$$

must be a symplectic form if G is extraspecial. Hence $n = 2m$, and the order of G is of the form p^{1+2m} for some positive integer m. One can be much more precise and prove that, for each given order p^{1+2m} there are precisely two extraspecial p-groups of that given order, up to isomorphism [Hup67, III, Sections 13, 13.7 and 13.8] or [Gor07, Chapter 5, 5.], but we do not need this detailed structure theory. We want to study the kernel of the 'inflation map'

$$K^G = \ker\left(\pi^*\colon H^*_{\mathrm{s}}(E, \mathbb{Z}/p) \to H^*_{\mathrm{s}}(G, \mathbb{Z}/p)\right).$$

This is a graded ideal in the graded ring $H^*_{\mathrm{s}}(E, \mathbb{Z}/p)$ (graded by cohomological degree). It is natural to expect that this should be, in general, generated by its degree-2 component or, even more precisely, by the class $\omega \in \mathrm{Hom}(\Lambda^2 E, \mathbb{Z}/p) = H^2_{\mathrm{s}}(E, \mathbb{Z}/p)$ given by the extension; see also Formula (3.1) in Section 3 for the description of the stable cohomology of abelian groups. In fact, Tezuka and Yagita in [TY11] studied a very similar problem in Section 9, p. 4492 and the following; see especially the problems they have mentioned on p. 4494, top and bottom, concerning what they cannot yet prove. Indeed, the expectation above is false in general (this is similar to the situation in ordinary group cohomology where conjectures that kernels of inflation maps associated with central extensions should always be the expected ones are false as well; see [Rus92, Proposition 9]). We show:

Theorem 1.4. *Let G be an extraspecial p-group of order p^{1+2m} as above. Then, provided $p > m$, the ideal K^G is generated by $\omega \in K^G_2$.*

Note that $\omega \in K^G_2$ always because it is the image, in $H^2_{\mathrm{s}}(E, \mathbb{Z}/p)$, of the class $\tilde{\omega} \in H^2(E, \mathbb{Z}/p)$, giving the central extension G, which vanishes when pulled back to $H^2(G, \mathbb{Z}/p)$ (the induced central extension of G has a section).

On the other hand:

Theorem 1.5. *Take $k = \mathbb{C}$. If G_0 is the extraspecial 2-group of order 2^{1+6} that is the preimage of the diagonal matrices* $\mathrm{diag}(\pm 1, \ldots, \pm 1)$ *in* $\mathrm{SO}_7(k)$ *under the natural covering map*

$$\mathrm{Spin}_7(k) \to \mathrm{SO}_7(k),$$

then K^{G_0} is not *generated by its degree-2 piece $K_2^{G_0} = \langle \omega \rangle$.*

This does not seem to be related to the fact that $p = 2$ is a special prime; we believe similar examples could very likely be given for every other prime p as well, as will become apparent from the construction in the proof.

Remark 1.6. Theorems 1.4 and 1.5 should be seen in the following context, which provided us with motivation for this work.

(1) As pointed out in [BT12, Theorem 11], the Bloch–Kato conjecture (Voevodsky's theorem) implies that, letting $\Gamma = \mathrm{Gal}(k(V/G))$ as above, and denoting

$$\Gamma^a = \Gamma/[\Gamma, \Gamma], \quad \Gamma^c = \Gamma/[\Gamma, [\Gamma, \Gamma]],$$

the natural map $H^*(\Gamma^a, \mathbb{Z}/p) \to H^*(\Gamma, \mathbb{Z}/p)$ is surjective, and its kernel K^{Γ^a} coincides with the kernel of $H^*(\Gamma^a, \mathbb{Z}/p) \to H^*(\Gamma^c, \mathbb{Z}/p)$ and is generated by its degree 2 component $K_2^{\Gamma^a}$ (note that since $[\Gamma, [\Gamma, \Gamma]] \subset [\Gamma, \Gamma]$, there is a natural homomorphism $\Gamma^c \to \Gamma^a$ giving $H^*(\Gamma^a, \mathbb{Z}/p) \to H^*(\Gamma^c, \mathbb{Z}/p)$); this follows not obviously from a spectral sequence argument, but in any case directly from the Bloch–Kato conjecture since for $L = k(V/G)$,

$$H^n(\Gamma^a, \mathbb{Z}/p) \simeq (L^* \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} L^*)/p, \quad H^n(\Gamma, \mathbb{Z}/p) \simeq K_n(L)/p,$$

and the Milnor K-group $K_n(L)$ is a quotient of $L^* \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} L^*$ by the nth graded piece of the ideal generated by the Steinberg relations in degree 2. Thus, whereas on the full profinite level, kernels of inflation maps are generated in degree 2, this property is not inherited by finite quotients of the full Galois group, that is, finite central extensions of finite abelian groups.

(2) The consequence of the Bloch–Kato conjecture in (1) shows the importance to understand central extensions of abelian groups for the computation of stable and unramified cohomology. In [BT17], after formula (1.2), the authors mention that for a finite central extension $0 \to \mathbb{Z}/p \to G^c \to G^a \to 0$ of an abelian group G^a, the kernel

of $H^*_{\mathrm{s}}(G^a, \mathbb{Z}/p) \to H^*_{\mathrm{s}}(G^c, \mathbb{Z}/p)$ *contains* the ideal generated by the kernel of $H^2_{\mathrm{s}}(G^a, \mathbb{Z}/p) \to H^2_{\mathrm{s}}(G^c, \mathbb{Z}/p)$; a preliminary version had *is equal to* instead of *contains*, and the present article shows that the containment can be strict contrary to what was expected.

2 Some Linear Algebra

We establish some results concerning the exterior algebra of a symplectic vector space over a field of any characteristic. Most of this is contained in [Bou05, Chapter VIII, Section 13, 3., pp. 203–210], but since the standing assumption there, Chapter VIII, is to work over a field of characteristic 0 where we are interested in the case of a base field of finite characteristic, it is necessary to point out in detail which statements go through unchanged and which ones require adaptation.

Let $\mathbb{F}$ be any field and V be a finite-dimensional $\mathbb{F}$-vector space of even dimension $n = 2m$. Suppose that V is symplectic, that is, endowed with a nondegenerate alternating bilinear form Ψ. Let $\mathrm{Sp}_{2m}(V, \Psi) = \mathrm{Sp}_{2m}$ be the corresponding symplectic group. From, for example [EKM08, Proposition 1.8], it follows that V is isometric to an orthogonal direct sum of m hyperbolic planes; in other words, there exists a symplectic basis

$$(e_1, \dots, e_m, e_{-m}, \dots, e_{-1}),$$

with $\Psi(e_i, e_j) = 0$ unless $i = -j$ when $\Psi(e_i, e_{-i}) = 1$. This is a statement entirely independent of the characteristic of $\mathbb{F}$, in particular, which also holds in characteristic two (the form is then at the same time alternating and symmetric). Let V^* be the dual vector space to V and (e_i^*) be the basis dual to the basis (e_i). We identify the alternating form Ψ with an element $\Gamma^* \in \Lambda^2 V^*$. Then

$$\Gamma^* = -\sum_{i=1}^{m} e_i^* \wedge e_{-i}^*.$$

Via the isomorphism $V \to V^*$ given by Ψ, the form Ψ induces a symplectic form Ψ^* on V^*. Identifying Ψ^* with an element Γ in $\Lambda^2 V$,

$$\Gamma = \sum_{i=1}^{m} e_i \wedge e_{-i}.$$

One also denotes by $X_- \colon \Lambda^* V \to \Lambda^* V$ the endomorphism induced by a left exterior product with Γ and by $X_+ \colon \Lambda^* V \to \Lambda^* V$ the endomorphism given by a left interior product (contraction) with $-\Gamma^*$; more precisely,

$$X_+(v_1 \wedge \cdots \wedge v_r) = \sum_{1 \le i < j \le r} (-\Psi)(v_i, v_j)(-1)^{i+j} v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_r.$$

Moreover, let $H\colon \Lambda^* V \to \Lambda^* V$ be the endomorphism that is multiplication by $(m-r)$ on $\Lambda^r V$ for $0 \le r \le 2m$. Then, as in [Bou05, p. 207, Example 19], it follows that

$$[X_+, X_-] = -H, \quad [H, X_+] = 2X_+, \quad [H, X_-] = -2X_- \tag{2.1}$$

so the vector subspace of $\mathrm{End}(\Lambda^* V)$ generated by X_+, X_-, H is a Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{F})$. Moreover, for the action of $\mathfrak{sl}(2, \mathbb{F})$ on $\Lambda^* V$, the subspace $\Lambda^r V$ is the subspace of elements of weight $m - r$.

In the following proposition and its proof, we make the conventions that for integers $i < 0$, $\Lambda^i V := 0$ and for binomial coefficients and positive integers n, $\binom{n}{i} := 0$.

Proposition 2.1. *Put $E_r = (\Lambda^r V) \cap \ker X_+$, the 'primitive elements' in $\Lambda^r V$. If $p = \mathrm{char}\, \mathbb{F} > \dim V/2 = m$ or $\mathrm{char}\, \mathbb{F} = 0$, then*

(1) for $r \le m-1$, the restriction of X_- to $\Lambda^r V$ is injective;
(2) for $r \ge m-1$, the restriction of X_- to $\Lambda^r V$ induces a surjection from $\Lambda^r V$ onto $\Lambda^{r+2} V$;
(3) for $r \le m$,

$$\Lambda^r V = E_r \oplus X_-(\Lambda^{r-2} V).$$

Moreover, E_r coincides with the submodule $F_r \subset \Lambda^r V$ defined as the span of all 'completely reducible' r-vectors $v_1 \wedge \cdots \wedge v_r$ such that $\langle v_1, \ldots, v_r \rangle$ is a totally isotropic subspace of V. Here completely reducible means simply a pure wedge product of the above form $v_1 \wedge \cdots \wedge v_r$.

Proof The proof is based on the following observations.

(I) Let E be any $\mathfrak{sl}(2, \mathbb{F})$-module and ϵ be a primitive element, by which we mean, as usual, $X_+(\epsilon) = 0$, and ϵ is an eigenvector for some $\lambda \in \mathbb{F}$ for H. Then, as long as ν is an integer such that $1 \le \nu < p$, it does make sense to define

$$\epsilon_\nu = \frac{(-1)^\nu}{\nu} X_-^\nu \epsilon, \quad \epsilon_0 = \epsilon, \quad \epsilon_{-1} = 0.$$

Then a straightforward computation with the relations (2.1), done in [Bou05, Chapter VIII, Section 1, 2., Proposition 1], shows that

$$H\epsilon_\nu = (\lambda - 2\bar{\nu})\epsilon_\nu, \qquad X_-\epsilon_\nu = -(\bar{\nu}+1)\epsilon_{\nu+1},$$
$$X_+\epsilon_\nu = (\lambda - \bar{\nu} + 1)\epsilon_{\nu-1}, \qquad (2.2)$$

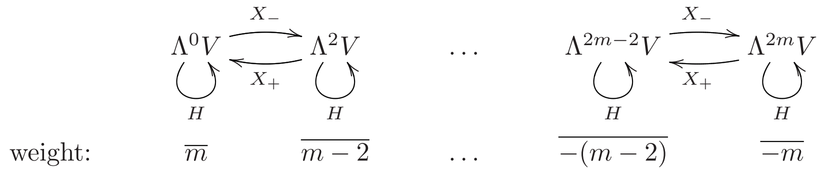
as long as all indices of the occurring ϵ's are $< p$. Here we put a bar on an integer to indicate that we consider it as an element of $\mathbb{F}$ via the natural homomorphism $\mathbb{Z} \to \mathbb{F}$, which for us will however be usually not injective.

(II) If we define F_r as in the statement of the proposition, then obviously $F_r \subset E_r$ and

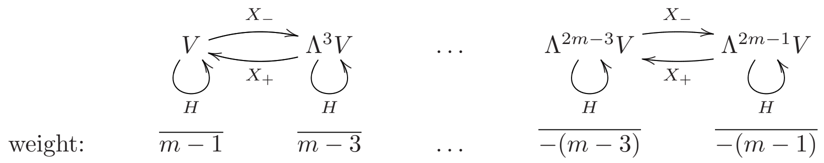
$$\dim F_r = \binom{2m}{r} - \binom{2m}{r-2}, \quad 0 \le r \le m.$$

This is proven in [DB09, Theorem 1.1] under no assumptions on $p = \operatorname{char} \mathbb{F}$.

For the module $E = \Lambda^* V$, we can thus display the action of the operators H, X_+, X_- schematically in the familiar way:



and



Now we start to use the assumption that $p = \operatorname{char} \mathbb{F} > m$.

If we start with a primitive element $\epsilon = \epsilon_0$ in one of the F_r, $0 \le r \le m$, of weight $\lambda = \overline{m-r}$ in $\{\bar{0}, \dots, \bar{m}\}$, then the ϵ_ν, as in item (I) above, are all defined for $\nu = 0, \dots, m$. Moreover, if μ is the largest integer such that $\epsilon_\mu \neq 0$, then $\mu \le m < p$ and μ can only possibly be equal to m if we start with ϵ_0 in F_0; excluding the latter case for a moment, we can use the third of (2.2) to obtain

$$0 = X_+(\epsilon_{\mu+1}) = (\lambda - \bar{\mu})\epsilon_\mu,$$

where now all indices are still $< p$, and one can only have that $\lambda - \bar{\mu} = 0$ in $\mathbb{F}$ if μ is the unique lift of λ in $\{0, \dots, m\}$. If $\mu = m$ and $\epsilon_0 \in F_0$, the