Cambridge University Press & Assessment
978-1-009-17970-6 — Optimal Mass Transport on Euclidean Spaces
Francesco Maggi
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OPTIMAL MASS TRANSPORT ON EUCLIDEAN SPACES

Over the past three decades, optimal mass transport has emerged as an active field with wide-ranging connections to the calculus of variations, partial differential equations (PDEs), and geometric analysis. This graduate-level introduction covers the field's theoretical foundation and key ideas in applications. By focusing on optimal mass transport problems in a Euclidean setting, the book is able to introduce concepts in a gradual, accessible way with minimal prerequisites, while remaining technically and conceptually complete. Working in a familiar context will help readers build geometric intuition quickly and give them a strong foundation in the subject. This book explores the relation between the Monge and Kantorovich transport problems, solving the former for both the linear transport cost (which is important in geometric applications) and the quadratic transport cost (which is central in PDE applications), starting from the solution of the latter for arbitrary transport costs.

Francesco Maggi is Professor of Mathematics at the University of Texas at Austin. His research interests include the calculus of variations, partial differential equations, and optimal mass transport. He is the author of *Sets of Finite Perimeter and Geometric Variational Problems: An Introduction to Geometric Measure Theory,* published by Cambridge University Press.

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FRANCESCO MAGGI
University of Texas at Austin

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Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment,
a department of the University of Cambridge.

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www.cambridge.org
Information on this title: www.cambridge.org/9781009179706

DOI: 10.1017/9781009179713

First published 2023

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging-in-Publication Data
Names: Maggi, Francesco, 1978– author.
Title: Optimal mass transport on Euclidean spaces / Francesco Maggi.
Description: Cambridge ; New York, NY : Cambridge University Press, 2023. |
Series: Cambridge studies in advanced mathematics ; 207 | Includes
bibliographical references and index.
Identifiers: LCCN 2023016425 | ISBN 9781009179706 (hardback) | ISBN
9781009179713 (ebook)
Subjects: LCSH: Transport theory – Mathematical models. | Mass transfer. |
Generalized spaces.
Classification: LCC QC175.2 .M34 2023 | DDC 530.13/8–dc23/eng/20230817
LC record available at https://lccn.loc.gov/2023016425

ISBN 978-1-009-17970-6 Hardback

"Francesco Maggi's book is a detailed and extremely well written explanation of the fascinating theory of Monge–Kantorovich optimal mass transfer. I especially recommend Part IV's discussion of the 'linear' cost problem and its subtle mathematical resolution."

– Lawrence C. Evans, UC Berkeley

"Over the last three decades, optimal transport has revolutionized the mathematical analysis of inequalities, differential equations, dynamical systems, and their applications to physics, economics, and computer science. By exposing the interplay between the discrete and Euclidean settings, Maggi's book makes this development uniquely accessible to advanced undergraduates and mathematical researchers with a minimum of prerequisites. It includes the first textbook accounts of the localization technique known as needle decomposition and its solution to Monge's centuries old cutting and filling problem (1781). This book will be an indispensable tool for advanced undergraduates and mathematical researchers alike."

– Robert McCann, University of Toronto

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Contents

Preface

In a hypothetical hierarchy of mathematical theories, the theory of optimal mass transport (OMT hereafter) lies at quite a fundamental level, yielding a formidable descriptive power in very general settings. The most striking example in this direction is the theory of curvature-dimension conditions, which exploits OMT to construct fine analytic and geometric tools in ambient spaces as general as metric spaces endowed with a measure. The Bourbakist aesthetics would thus demand OMT to be presented in the greatest possible generality from the onset, narrowing the scope of the theory only when strictly necessary. In contrast, this book stems from the pedagogically more pragmatic viewpoint that many key features of OMT (and of its applications) already appear in full focus when working in the simplest ambient space, the Euclidean space $\mathbb{R}^n$, and with the simplest transport costs per unit mass, namely, the "linear" transport cost $c(x,y) = |x-y|$ and the quadratic transport cost $c(x,y) = |x-y|^2$. Readers of this book, who are assumed to be graduate students with an interest in Analysis, should find in these pages sufficient background to start working on research problems involving OMT – especially those involving partial differential equations (PDEs); at the same time, having mastered the basics of the theory in its most intuitive and grounded setting, they should be in an excellent position to study more complete and general accounts on OMT, like those contained in the monographs [Vil03, Vil09, San15, AGS08]. For other introductory treatments that could serve well to the same purpose, see, for example [ABS21, FG21].

The story of OMT began in 1781, that is, in the midst of a founding period for the fields of Analysis and PDE, with the formulation of a *transport problem* by Monge. Examples of famous problems formulated roughly at the same time include the wave equation (1746), the Euler equations (1757), Plateau's problem[1] and the minimal surface equation (1760), and the heat equation and the Navier–Stokes equations (1822). An interesting common trait of all these problems is that the frameworks in which they had been originally formulated have all proved inadequate for their satisfactory solutions. For example, the study of heat and wave equations has stimulated the investigation of Fourier's series, with the corresponding development of functional and harmonic analysis, and of the notion of *distributional* solution. Similarly, the study of the minimal surface equation and of the Plateau's problem has inspired a profound revision of the notion of surface itself, leading to the development of geometric measure theory. The Monge problem has a similar history:[2] an original formulation (essentially intractable), and a modern reformulation, proposed by Kantorovich in the 1940s, which leads to a broader class of problems and to many new questions. The main theme of this book is exploring the relation between the Monge and Kantorovich transport problems, solving the former both for the linear transport cost (the one originally considered by Monge, which is of great importance in geometric applications) and for the quadratic transport cost (which is central in applications to PDE), starting from the solution of the latter for arbitrary transport costs.

The book is divided in four parts, requiring increasing levels of mathematical maturity and technical proficiency at the reader's end. Besides a prerequisite[3] familiarity with the basic theory of Radon measures in $\mathbb{R}^n$, the book is essentially self-contained.

Part I opens with an introduction to the original minimization problem formulated by Monge in terms of **transport maps** and includes a discussion about the intractability of the Monge problem by a direct approach, as well as some basic examples of (sometimes optimal) transport maps (Chapter 1). It then moves to the solution of the **discrete** OMT problem with a generic transport cost $c(x,y)$ (Chapter 2), which serves to introduce in a natural way three key ideas behind Kantorovich's approach to OMT problems: the notions of **transport plan**, ***c*-cyclical monotonicity**, and **Kantorovich duality**. Kantorovich's theory is then presented in Chapter 3, leading to existence and characterization results for optimal transport plans with respect to generic transport costs.

1 The minimization of surface area under a prescribed boundary condition (together with the related minimal surface equation) has been studied by mathematicians at least since the work of Lagrange (1760). The modern consolidated terminology calls this minimization problem "Plateau's problem," although Plateau's contribution was dated almost a century later (1849) and consisted in extensive experimental work on soap films.

2 For a complete and accurate account on the history of the Monge problem and on the development of OMT, see the bibliographical notes of Villani's treatise [Vil09].

3 All the relevant terminologies, notations, and required results are summarized in Appendix A, which is for the most part a synopsis of [Mag12, Part I].

The optimal transport *plans* constructed in Kantorovich's theory are more general and flexible objects than the optimal transport *maps* sought by Monge, which explains why solving the Kantorovich problem is way easier than solving the Monge problem. Moreover, a transport map canonically induces a transport plan with the same transport cost, thus leading to the fundamental question: *When are optimal transport plans induced by optimal transport maps?* Parts II and IV provide answers to these questions in the cases of the quadratic and linear transport costs, respectively.

Part II opens with the **Brenier theorem** (Chapter 4), which asserts the existence of an optimal transport map in the Monge problem with quadratic cost under the assumptions that the origin mass distribution is absolutely continuous with respect to the Lebesgue measure and that both the origin and final mass distributions have finite second order moments; moreover, this optimal transport map comes in the form of the *gradient of a convex function*, and is *uniquely* determined, and therefore called *the* **Brenier map** from the origin to the final mass distribution. In Chapter 5, we establish some sharp results on the first order differentiability of convex functions, which we then use in Chapter 6 to prove McCann's remarkable extension of the Brenier theorem – in which the absolute continuity assumption on the origin mass distribution is sharply weakened, and the finiteness of second order moments is entirely dropped. In both the Brenier theorem and the Brenier–McCann theorem, the transport condition is expressed in a measure-theoretic form (see (1.6) in Chapter 1) which is weaker than the "infinitesimal transport condition" originally envisioned by Monge (see (1.1) in Chapter 1). The former implies the latter for transport maps that are Lipschitz continuous and injective, but, unfortunately, both properties are not generally valid for gradients of convex functions. To address this point, in Chapter 7, we provide a detailed analysis of the second order differentiability properties of convex functions, which we then exploit in Chapter 8 to prove the validity of the Monge–Ampère equation for Brenier maps between absolutely continuous distributions of mass. In turn, the latter result is of key technical importance for the applications of quadratic OMT problems to PDE and geometric/functional inequalities.

Part III has two main themes: the first one describes some celebrated applications of Brenier maps to mathematical models of physical interests; the second one introduces the geometric structure of the **Wasserstein space**. That is the space of finite second-moment probability measures $\mathcal{P}_2(\mathbb{R}^n)$ endowed with the distance $\mathbf{W}_2$ defined by taking the square root of the minimum value in the Kantorovich problem with quadratic cost. The close relation between the purely geometrical properties of the Wasserstein space and the inner workings of many mathematical models of basic physical importance is one of the most charming and inspiring traits of OMT theory and definitely the reason why OMT is so relevant for mathematicians with such largely different backgrounds.

We begin the exposition of these ideas in Chapter 9, where we present OMT proofs of two inequalities of paramount geometric and physical importance, namely, the Euclidean isoperimetric inequality and the Sobolev inequality. We then continue in Chapter 10 with the analysis of a model for self-interacting gases at equilibrium. While studying the uniqueness problem for minimizers in this model, we naturally introduce an OMT-based notion of convex combination between probability measures, known as **displacement interpolation**, together with a corresponding class of **displacement convex** "internal energies." The latter include an example of paramount physical importance, namely, the (negative) entropy $\mathcal{S}$ of a gas. As a further application of displacement convexity (beyond the uniqueness of equilibria of self-interacting gases), we close Chapter 10 with an OMT proof of another key geometric inequality, the Brunn–Minkowski inequality.

In Chapter 11, we introduce the Wasserstein space $(\mathcal{P}_2(\mathbb{R}^n), \mathbf{W}_2)$, build up some geometric intuition on it by a series of examples, prove that it is a complete metric space, and explain how to interpret displacement convexity as *geodesic interpolation* in $(\mathcal{P}_2(\mathbb{R}^n), \mathbf{W}_2)$. We then move, in the two subsequent chapters, to illustrate how to interpret many parabolic PDEs as *gradient flows* of displacement convex energies in the Wasserstein space. In Chapter 12, we introduce the notion of gradient flow, discuss why interpreting an evolution equation as a gradient flow is useful, how it is possible that the same evolution equation may be seen as the gradient flow of different energies, and how to construct gradient flows through the **minimizing movements scheme**. Then, in Chapter 13, we exploit the minimizing movements scheme framework to prove that the Fokker–Planck equation (describing the motion of a particle under the action of a chemical potential and of white noise forces due to molecular collisions) can be characterized (when the chemical potential is convex) as the gradient flow of a displacement convex functional on the Wasserstein space, and, as a further application, we derive quantitative rates for convergence to equilibrium in the Fokker–Planck equation.

In Chapter 14, we obtain additional insights about the geometry of the Wasserstein space by looking at the **Euler equations** for the motion of an incompressible fluid. The Euler equations describe the motion of an incompressible fluid in the absence of friction/viscosity and can be characterized as geodesics equations in the (infinite-dimensional) "manifold" $\mathcal{M}$ of volume-preserving transformations of a domain. At the same time, geodesics (on a manifold embedded in some Euclidean space) can be characterized by a limiting procedure involving an increasing number of "mid-point projections" in the ambient space: there lies the connection with OMT, since the Brenier theorem allows us to characterize L^2-projections over $\mathcal{M}$ as compositions with Brenier maps. The analysis of the Euler equations serves also to introduce two crucial objects: the **action functional** of an incompressible fluid (time integral of the total kinetic energy) and the **continuity equation** (describing how the Eulerian velocity of the fluid transports its mass density). In Chapter 15, we refer to these objects when formally introducing the concepts of **(Eulerian) velocity of a curve of measures** in $\mathcal{P}_2(\mathbb{R}^n)$ and characterize the Wasserstein distance between the end points of such a curve in terms of minimization of a corresponding action functional. This is the celebrated **Benamou–Brenier formula**, which provides the entry point to understand the “Riemannian” (or “infinitesimally Hilbertian”) structure of the Wasserstein space. We only briefly explore the latter direction, by quickly reviewing the notion of gradient induced by such Riemannian structure (Otto’s calculus).

Part IV begins with Chapter 16, where a sharp result for the existence of optimal transport maps in dimension one is presented. In Chapter 17, we first introduce the fundamental *disintegration theorem* and then exploit it to give a useful geometric characterization of transport plans induced by transport maps and to prove the equivalence of infima for the Monge and Kantorovich problems when the origin mass distribution is atomless. We then move, in Chapter 18, to construct optimal transport maps for the Monge problem with linear transport cost. We do so by implementing a celebrated argument due to Sudakov, which exploits disintegration theory to reduce the construction of an optimal transport map to the solution of a family of one-dimensional transport problems. The generalization of Sudakov’s argument to more general ambient spaces (like Riemannian manifolds or even metric measure spaces) lies at the heart of a powerful method for proving geometric and functional inequalities, known as the “needle decomposition method,” and usually formalized in the literature as a “localization theorem.” In Chapter 19, we present these ideas in the Euclidean setting. Although this restricted setting does not allow us to present the most interesting applications of the technique itself, its discussion seems, however, sufficient to illustrate several key aspects of the method, thus putting readers in an ideal position to undertake further reading on this important subject.

Having put the focus on the clarity of the mathematical exposition above anything else, the main body of this book contains very few bibliographical references and almost no bibliographical digressions. A set of bibliographical notes has been included in the appendix with the main intent of acknowledging the original papers and references used in the preparation of the book and of pointing students to a few of the many possible further readings.

For comprehensive bibliographies and historical notes on OMT, we refer to [AGS08, Vil09, San15].

This book originates from a short course on OMT I taught at the Universidad Autónoma de Madrid in 2015 at the invitation of Matteo Bonforte, Daniel Faraco, and Juan Luis Vázquez. An expanded version of the lecture notes of that short course formed the initial core of a graduate course I taught at the University of Texas at Austin during the fall of 2020, and whose contents roughly correspond to the first 14 chapters of this book. The remaining chapters have been written in a more advanced style and without the precious feedback generated from class teaching. For this reason, I am very grateful to Fabio Cavalletti, Nicola Gigli, Carlo Nitsch, and Aldo Pratelli who, in reading those final four chapters, have provided me with very insightful comments, spotted subtle problems, and suggested possible solutions that led to some major revisions (and, I think, eventually, to a very nice presentation of some deep and beautiful results!). I would also like to thank Lorenzo Brasco, Kenneth DeMason, Luigi De Pascale, Andrea Mondino, Robin Neumayer, Daniel Restrepo, Filippo Santambrogio, and Daniele Semola for providing me with additional useful comments that improved the correctness and clarity of the text. Finally, I thank Alessio Figalli for his initial encouragement in turning my lecture notes into a book.

With my gratitude to Luigi Ambrosio and Cedric Villani, from whom I have first learned OMT about 20 years ago during courses at the Scuola Normale Superiore di Pisa and at the Mathematisches Forschungsinstitut Oberwolfach, and with my sincere admiration for the many colleagues who have contributed to the discovery of the incredibly beautiful Mathematics contained in this book, I wish readers to find here plenty of motivations, insights, and enjoyment while learning about OMT and preparing themselves for contributing to this theory with their future discoveries!

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More Information

Notation

a.e., almost everywhere
s.t., such that
w.r.t., with respect to
$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, natural, integer, rational, and real numbers
$\mathbb{R}^n$, the n-dimensional Euclidean space
$B_r(x)$, open ball in $\mathbb{R}^n$ with center x and radius r (Euclidean metric)
$\mathrm{Int}(E)$, interior of a set $E \subset \mathbb{R}^n$ (Euclidean topology)
$\mathrm{Cl}(E)$, closure of a set $E \subset \mathbb{R}^n$ (Euclidean topology)
∂E, boundary of a set $E \subset \mathbb{R}^n$ (Euclidean topology)
$\mathbb{S}^n$, the n-dimensional sphere in $\mathbb{R}^{n+1}$
ν_E, the outer unit normal $\nu_E : \partial E \to \mathbb{S}^{n-1}$ of a set $E \subset \mathbb{R}^n$ with C^1-boundary
$\mathbb{R}^{n\times m}$, matrices with n-rows and m-columns
A^*, the transpose of a matrix/linear operator
$\mathbb{R}^{n\times n}_{\mathrm{sym}}$, matrices $A \in \mathbb{R}^{n\times n}$, with $A = A^*$
$w \otimes v$ ($v \in \mathbb{R}^n$, $w \in \mathbb{R}^m$), the linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$
defined by $(w \otimes v)[e] = (v \cdot e)\, w$
$\mathcal{B}(\mathbb{R}^n)$, the Borel subsets of $\mathbb{R}^n$
$\mathcal{L}^n$, the Lebesgue measure on $\mathbb{R}^n$
$\mathcal{H}^k$, the k-dimensional Hausdorff measure on $\mathbb{R}^n$
$\mathcal{P}(\mathbb{R}^n)$, probability measures on $\mathbb{R}^n$
$\mathcal{P}_{\mathrm{ac}}(\mathbb{R}^n)$, measures in $\mathcal{P}(\mathbb{R}^n)$ that are absolutely continuous w.r.t. $\mathcal{L}^n$
$\mathcal{P}_p(\mathbb{R}^n)$, measures in $\mathcal{P}(\mathbb{R}^n)$ with finite p-moment ($1 \le p < \infty$)
$\mathcal{P}_{p,\mathrm{ac}}(\mathbb{R}^n) = \mathcal{P}_{\mathrm{ac}}(\mathbb{R}^n) \cap \mathcal{P}_p(\mathbb{R}^n)$
$\mu \ll \nu$, μ is absolutely continuous w.r.t. ν (μ, ν measures on $\mathbb{R}^n$)
$L^p(\mathbb{R}^n)$, p-summable functions w.r.t. $\mathcal{L}^n$ ($1 \le p \le \infty$)
$L^p(\mu)$, p-summable functions w.r.t. a Borel measure μ ($1 \le p \le \infty$)
$C^0_c(\mathbb{R}^n)$, functions on $\mathbb{R}^n$ that are continuous with compact support

$C_b^0(\mathbb{R}^n)$, functions on $\mathbb{R}^n$ that are continuous and bounded
$\overset{*}{\rightharpoonup}$, weak-star convergence of Radon measures (C_c^0-test functions)
$\overset{n}{\rightharpoonup}$, narrow convergence of Radon measures (C_b^0-test functions)
$C_c^0(\mathbb{R}^n) \otimes C_c^0(\mathbb{R}^m)$, functions $f(x,y) = g(x)\,h(y)$, $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$,
with $g \in C_c^0(\mathbb{R}^n)$, $h \in C_c^0(\mathbb{R}^m)$
$C^{0,\alpha}(\mathbb{R}^n)$, α-Hölder continuous functions ($\alpha \in (0,1]$, $C^{0,1} = \mathrm{Lip}$)
$C^{k,\alpha}(\mathbb{R}^n)$, k-times differentiable functions whose kth gradient is in $C^{0,\alpha}$
Df, distributional derivative of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$
∇f, pointwise gradient of f or density of Df w.r.t. $\mathcal{L}^n$
$f\,d\mu$, or $f(x)\,d\mu(x)$, the measure defined
by the integral of $f \in L^1_{\mathrm{loc}}(\mu)$ w.r.t. μ
$\int_X f(x,y)\,d\mu(x)$, integration w.r.t. μ occurs in the x-variable of $f(x,y)$
$f|_E$, the restriction of f to $E \subset F$ when f is a function defined on F
$\mathbf{p} : X \times Y \to X$, projection on the first factor of $X \times Y$, i.e., $\mathbf{p}(x,y) = x$
$\mathbf{q} : X \times Y \to Y$, projection on the second factor of $X \times Y$, i.e., $\mathbf{q}(x,y) = y$
$\mathbf{id}_X$, the identity map on the set X (X omitted if clear from the context)
$C(a,b,\ldots)$, a generic constant depending only $a,b,\ldots$
whose value may increase at each subsequent appearance

Disambiguation: The terms "formal" and "formally" are used in this book to indicate the quality of being endowed with full mathematical rigor. This may create confusion since, in the Analysis literature, the term "formal" is sometimes (if not often) used to express the quality of "being presented without a full justification": e.g., expressions like "by a formal integration by parts, taken without discussing the negligibility of boundary terms" or "by a formal argument that does not take into account measurability issues" are quite common in the OMT literature. However, synonyms of "formal" are "official," "legal," "validated," and "authoritative," which definitely point to the quality of possessing full mathematical rigor; hence, the use of "formal" in this book.