Preface

Dear reader,

By reading this Element you are making an eternal mark on the history of the world, both in its past and in its future. Indeed:

i) It has always in the past been the case that you would be reading this Element (sometime in the future of that past).

ii) It will always in the future be the case that you had been reading this Element (sometime in the past of that future).

These two statements are simple, but important, patterns of *temporal reasoning*, which is the subject of this Element. More precisely, the Element is about *temporal logical reasoning* based on various formal systems of *temporal logics*.

This concise exposition is intended to provide some philosophical insights and discussions on the role and applications of formal logic to temporal reasoning, as well as some technical details, including some illustrative proofs, of the semantic and deductive aspects of these applications. It was a challenging task to strike a good balance between these within the space limitations, but I hope that readers interested in either of these aspects of temporal logics will find enough of interest and value in this Element.

Although the exposition is mostly on a basic level, a beneficial reading of the Element still assumes the reader has some background in formal classical logic, as well as in the basics of propositional modal logics, say, within the first chapters of Hodges (2001), Halbach (2010), or Goranko (2016) for classical logic, and of Fitting and Mendelsohn (1998), Blackburn, de Rijke, and Venema (2001), van Benthem (2010), and Uckelman (in press), on modal logic.

Due to space limitations, I have made few references to the vast relevant literature on temporal logics throughout this Element. This is partly compensated for by the additional references at the end of each section. In addition, key general references on philosophical and technical aspects of temporal logics and on their applications include: Prior (1957, 1967, 1968), Rescher and Urquhart (1971), McArthur (1976), van Benthem (1983), Gabbay, Hodkinson, and Reynolds (1994), van Benthem (1995), Øhrstrøm and Hasle (1995), Gabbay, Reynolds, and Finger (2000), Venema (2001), all chapters in Gabbay and Guenthner (2002), Fisher, Gabbay, and Vila (2005), Demri, Goranko, and Lange (2016), Goranko and Rumberg (2020), and the bibliographies therein.

The undefined notation used in this Element is assumed common or self-explanatory, including the use of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ to denote, respectively, the sets of natural numbers, integers, rationals, and reals.

1 Temporal Reasoning and Logics: Introduction and a Brief Historical Overview

What is temporal logic? This is the first question that springs to mind for the uninitiated reader seeing this Element. While the Element itself aims to give the ultimate answer, I will begin with a very brief and sketchy historical overview on temporal reasoning and the origins of temporal logic.

1.1 What Is Time?

Temporal logics are logics for *reasoning about Time*. So, the question to ask before looking further is *what is Time*? A classic quote by Augustine of Hippo (aka Saint Augustine, AD 354-430) says:

> *What then is time?*
> *If no one asks me, I know what it is.*
> *If I wish to explain it to him who asks, I do not know.*

I am afraid we are not much wiser today on that question. Indeed, various witty, but mostly superficial, answers have been proposed, such as:

> *Time is nature's way to keep everything from happening all at once.*

> *Time is the sequence of the events happening in the universe.*

or just:

> *Time is what clocks measure.*

None of these are really satisfactory, especially after the insights on the relativity of time that have come from twentieth-century physics and cosmology. Apparently, understanding time takes time …. So, I will simply take it as a primitive concept on which the reader has some initial intuition. That intuition will be refined, enriched, and also challenged in this Element.

1.2 What Is Temporal Logic?

This is a much easier question. The most concise answer is that, whereas classical logic reasons about a snapshot of the universe, where everything is fixed (also in truth value), temporal logic reasons about the temporal dynamics of the universe, that is, about what is happening as time passes and how that affects the truth of propositions about the world. Thus, temporal logic is not so much about reasoning *about time*, but rather about reasoning about events happening *in time*. More precisely, temporal logic helps us formalise and conduct reasoning with *temporal propositions*, such as the following:

- You are reading (*now*).
- *Sometime in the past* you were not reading.
- Sometime in the future you will not be reading.
- You will be reading *sometime in the future* and *never in the future* thereafter.
- *Always in the past* you were going to be reading (*sometime in the future of that past*).
- *Always (in the future)* you will have been reading (*in the past of that future*).
- It will (*sometime in the future*) be the case that you have been reading (*for a while*) but that you are not reading *any more*.
- *Sometime in the past* you were not reading but it had *always before* been the case that you were going to be reading (*sometime in the future of that past*).
- You have been reading *since sometime in the past* and will go on reading *until sometime in the future*, and so on.

Suppose you are wondering, for instance, whether

> the fact that *you are reading this book now, but were not reading it a year ago, logically implies that you have been reading the book since sometime in the past when you had never been reading it before, but you would be reading it until some future time when you will not be reading it ever again.*

Then, you are doing the kind of temporal logical reasoning which this Element is about. And, if you wish to know the answer, it is: *it depends*! It depends on some ontological assumptions about the nature of time, on the formal logical language used, and how exactly the query above is formalised in it, as well as on the precise, formal logical semantics which you adopt for defining the notion of logical consequence to which that query refers. So, in fact, as we will see further, there is not just one and only one *temporal logic* that covers all our temporal logical reasoning, but a rich variety of many systems of *temporal logics*, suited for reasoning under different assumptions, in different formal languages, and for different formal semantics adopted for them. This Element aims at providing a panoramic view on the landscape of these temporal logics.

1.3 Origins and Antiquity: Zeno's Paradoxes and Sea-Battles

Temporal reasoning has been an essential aspect of human reasoning ever since humans developed the concept of time. While we do not seem to know when exactly that happened, the discussion on temporality and reasoning about time goes back to antiquity, and examples can be found even in the Bible. Then, in Ancient Greece we find Zeno's arguments referring to the apparently paradoxical nature of time manifested by the infinite divisibility of time (and space) intervals. Zeno's paradoxes 'The Dichotomy', 'Achilles and the

Tortoise', and 'The Arrow' challenge our concept of time and its properties, and the closely related notions of space, motion, and change.

Perhaps the earliest more explicit reference to logical aspects of temporal reasoning, however, is Aristotle's argument in *De Interpretatione* Aristotle (1984/350, ch. 9) that definite truth-values cannot, at the present time, be ascribed to *future contingents*, that is, to statements about future events which may or may not occur, such as '*There will be a sea-battle tomorrow*.'

1.4 Time, Necessity, and Determinism: Diodorus Cronus' Master Argument

Just a few decades after Aristotle, the philosopher Diodorus Cronus from the Megarian school demonstrated the problem with future contingents in his famous Master Argument, based on the following three propositions:

(D1) *Every proposition which is true about the past is necessarily true.*
(D2) *An impossible proposition cannot follow*[1] *from a possible one.*
(D3) *There is a proposition which is possible, but which neither is, nor will be true.*

Diodorus argued[2] that these cannot all be true together, yet (D1) and (D2) should be accepted as true. Consequently, (D3) must be rejected. Therefore, 'possible' can be defined as '*that which is true or will ever become true*' and, correspondingly, 'necessary' is '*that which is true and will always be true*'. Diodorus' argument has been regarded as supporting determinism, even fatalism.

I will come back to the Master Argument in Section 5, where I will present Prior's formal reconstruction of that argument and his proposed formal logical solutions leading to the birth of logics of branching time.

1.5 Medieval Times: Determinism versus Free Will

During the Middle Ages there were heated philosophical and theological debates on free will vs. determinism. The problem at the heart of these debates was how to reconcile God's foreknowledge of a person's future decisions and actions, suggesting that the evolution of the world is predetermined, with the that person's free will and moral accountability for their decisions and actions.

[1] Note the possible ambiguity: 'follow' can be read in a temporal or in a logical sense. Apparently, Diodorus meant the latter, in the sense of Lewis' strict implication.
[2] Unfortunately, the original version of the argument itself has not been preserved.

Notably, the thirteenth-century scholastic philosopher and theologian William of Ockham held that propositions about the contingent future cannot be known in advance by humans as true or false, but *humans have a freedom of choice between different possible futures, even though God – being independent of time and beyond it – already knows that possible future that will actually take place.* This position suggests the idea of a *tree-like, forward-branching model of time*, where the past is fixed and cannot be changed, but there are many possible futures, hence many timelines (histories), of which just one will *actually* take place (the idea of the 'Thin Red Line', which can also be traced in the works of the sixteenth-century Jesuit priest and scholar Luis de Molina). Furthermore, the truth of statements about the future is relativised to the possible future that is presumed to be the actual one. This model of time, now often called *Ockhamist*, gives rise to Prior's *Ockhamist semantics* of branching time logic, presented in Section 7.

Another landmark in the medieval history of temporal logic is Avicenna's (Ibn Sina's) extension of Aristotle's syllogisms with temporal aspects, such as 'All A are always B', 'All A are at some time B', 'Some A are never B', and so on.

1.6 Precursors to Temporal Logic

Various arguments and patterns of reasoning about temporality, related to non-determinism, historical necessity, humans' free will, God's will and knowledge, and so on, were proposed in post-medieval times and until the twentieth century. Some such temporal arguments can be found, inter alia, in the works of Boole, Hamilton, Bergson, and most notably Peirce, who disagreed with the view, prevailing then amongst philosophers, that time is an 'extra-logical matter' and argued in favour of logic-based treatment of time and temporality. However, Peirce objected to the idea that future contingents can currently have definite truth values, as he argued that there is no '*actual future*' yet at present, but only many such *possible futures*. Therefore, according to him, truth in the future should mean truth in *all* possible futures. Later, his ideas led Prior to introduce one of his main systems of branching time temporal logics which he called 'Peircean', presented in Section 6.

Much more happened in philosophy and other sciences during the early-mid twentieth century which paved the way to the emergence of formal temporal logic. Here are some landmarks:

- In 1908, Hermann Minkowski gave a public lecture on 'Space and Time' where he presented the ideas of relativity of time and its close relationship with space, eventually leading to what is now called '*Minkowski's*

4-dimensional spacetime'. Later in the twentieth century, several formal logical systems were developed, purporting to formalise and axiomatise Minkowski's spacetime.

- Again in 1908, John McTaggart, driven by the idea to demonstrate the 'unreality of time', proposed two alternative approaches to modelling time, now known as *McTaggart's time series*. See Section 3.4 on these and their relation with formal logical reasoning about time.
- In 1920, motivated by attempts to resolve Aristotle's 'sea-battle tomorrow' puzzle and the related problems with future contingents, Jan Łukasiewicz proposed a *three-valued logic*, assigning to future contingent statements the new, third truth value of 'undetermined'.
- In 1947, Hans Reichenbach, following some ideas of Otto Jespersen, developed his very influential *theory of tenses*, where he characterised most of the tenses in natural language by using a triple of time points related to the utterance of tensed statements, namely: the *speech time* S, the *reference time* R, and the *event time* E. For more on that, see Section 9.5.3 and the bibliographic notes.
- Again in 1947, Jerzy Łoś presented a 'positional calculus' intended to formalise 'Mill's canons', in which he used certain temporal functions. His system is regarded by some as one of the prime precursors of modern temporal logic.
- Several other twentieth-century philosophers, including Bertrand Russell, John Findlay, and Charles Hamblin, have also provided important insights leading to the creation of temporal logic; see some references at the end of this section.

1.7 The Birth of Temporal Logic: Prior and Post-Prior

In the 1940s the philosopher and logician Arthur Prior became strongly interested in philosophical and theological problems related to determinism and divine foreknowledge versus indeterminism, human free will, and moral responsibility. In that context, Prior set out to analyse, formalise, and eventually try to resolve some famous problems and arguments from antiquity, including Aristotle's 'sea-battle tomorrow' problem and Diodorus Cronus' Master Argument. Besides, he also wanted to develop a logical theory of tenses. That led him to the invention of several formal systems of what he then called 'tense logic', several of which are presented and discussed here. As acknowledged by himself, Prior's seminal work was influenced by some important precursors mentioned earlier, including Findlay (whom Prior regarded as the founding father of temporal logic), Reichenbach, Łukasiewicz, and Łoś. Prior's work initiated the modern epoch of temporal logical reasoning, which found numerous important applications not only in philosophy, but also in computer science, artificial intelligence, and linguistics, briefly discussed in Section 9.5.

Some References

For further readings on the history of temporal reasoning and logics see Prior (1957), Prior (1967), Rescher and Urquhart (1971, ch. XVII), Øhrstrøm and Hasle (1995), Øhrstrøm and Hasle (2006), Meyer (2013), Meyer (2015), Hodges and Johnston (2017), Øhrstrøm (2019), Øhrstrøm and Hasle (2020). For more on Prior's philosophical views on time, see also Hasle, Blackburn, and Øhrstrøm (2017), Blackburn, Hasle, and Øhrstrøm (2019).

2 The Variety of Models of Time

What is the right model of the flow of time? Is it unique, or are there many? What properties does time have? Is it discrete or dense, continuous or gappy? Is it linear, or branching into the future? Does time have a beginning or an end? Or is it not circular, as our watches and calendars suggest? Further, what are the primitive entities in the structure of time – time instants, or time intervals, or something else?

These fundamental ontological and philosophical questions about the nature of time do not have definitive and unique answers, but they rather lead to a rich variety of formal models of time, and respectively to a variety of temporal logics, including *logics for linear time and for branching time*, *logics for discrete time and for dense time*, *point-based logics and interval-based logics*, and so on. Before exploring these logics, let us first look at the two most basic types of formal models of time with respect to the temporal entities which they adopt as primitives: *instant-based* and *interval-based* models.

2.1 Instant-Based Models of Time and Their Properties

The primitive entities in instant-based models of time are *points in time*, usually called **(time) instants**, or **moments**. The basic relationship between them, besides equality, is **temporal precedence**. Thus, an **instant-based model of time** is a structure of the type $\mathcal{T} = \langle T, \prec \rangle$, consisting of a non-empty set of instants T with a binary relation $\prec$ of precedence on it. Sometimes, the preference relation will be given as $\preceq$, where $x \preceq y$ is an abbreviation of $x \prec y \vee x = y$. If $s, t \in \mathcal{T}$ and $s \prec t$, then we say that s **precedes** t, or that s **is a predecessor of** t, and respectively that t **succeeds** s, or t **is a successor of** s.

Some natural properties can be imposed on the precedence relation in instant-based models of time. Most (but not all) such properties can be expressed by sentences of classical first-order logic for instant-based models, as follows:

- **reflexivity** (every instant precedes itself): $\forall x(x \prec x)$.
- **irreflexivity** (no instant precedes itself): $\forall x \neg(x \prec x)$.
  I hereafter assume that $\prec$ is irreflexive,[3] unless otherwise stated.
- **transitivity**: $\forall x \forall y \forall z(x \prec y \land y \prec z \rightarrow x \prec z)$.
  An irreflexive and transitive relation is called a **strict partial ordering**. A reflexive and transitive relation is called a **(non-strict) partial pre-ordering**.
- **asymmetry** (two instants cannot precede each other):
  $\forall x \forall y \neg(x \prec y \land y \prec x)$.
  Note that every strict partial ordering is asymmetric.
- **anti-symmetry** (if two instants precede each other, then they are identical):
  $\forall x \forall y(x \prec y \land y \prec x \rightarrow x = y)$.
  An anti-symmetric partial pre-ordering is called a **partial ordering**.
- **trichotomy** (every two instants are *comparable*, i.e., they are either identical, or one precedes the other): $\forall x \forall y(x = y \lor x \prec y \lor y \prec x)$.
  This property is also known as **connectedness**, or **linearity**.
  A strict (respectively, non-strict) partial ordering which satisfies trichotomy is a **strict linear ordering** (respectively, **non-strict linear ordering**).
- **forward-connectedness**, aka **forward-linearity** (every two instants which are preceded by the same instant are comparable):
  $\forall x \forall y \forall z(z \prec x \land z \prec y \rightarrow (x = y \lor x \prec y \lor y \prec x))$.
- **backward-connectedness**, aka **backward-linearity** (every two instants which precede the same instant are comparable):
  $\forall x \forall y \forall z(x \prec z \land y \prec z \rightarrow (x = y \lor x \prec y \lor y \prec x))$.
- **existence of a beginning**: $\exists x \neg \exists y(y \prec x)$.
- **existence of an end**: $\exists x \neg \exists y(x \prec y)$.
- **no beginning**: $\forall x \exists y(y \prec x)$.
- **no end (unboundedness)**: $\forall x \exists y(x \prec y)$.
- **density** (between every two instants, of which one precedes the other, there is an instant): $\forall x \forall y(x \prec y \rightarrow \exists z(x \prec z \land z \prec y))$;
- **forward-discreteness** (every non-last instant has an immediate successor):
  $\forall x \forall y(x \prec y \rightarrow \exists z(x \prec z \land z \leq y \land \neg \exists u(x \prec u \land u \prec z)))$;

[3] This is not an ontological assumption, just a matter of convention on the time precedence.

- **backward-discreteness** (every non-first instant has an immediate predecessor): $\forall x \forall y(y \prec x \rightarrow \exists z(z \prec x \wedge y \leq z \wedge \neg \exists u(z \prec u \wedge u \prec x)))$.

A (instant-based) model of time $\mathcal{T} = \langle T, \prec \rangle$, is (strictly) **linear** if $\prec$ is a (strict) linear ordering.

Note that, in linear models, the two discreteness conditions simplify to

- $\forall x \forall y(x \prec y \rightarrow \exists z(x \prec z \wedge \forall u(x \prec u \rightarrow z \leq u)))$ and, respectively:
- $\forall x \forall y(y \prec x \rightarrow \exists z(z \prec x \wedge \forall u(u \prec x \rightarrow u \leq z)))$.

A model of time $\mathcal{T} = \langle T, \prec \rangle$, is **tree-like**, or **forward-branching** if $\prec$ is a backward-linear partial ordering, that is, a partial ordering in which every instant has the set of its predecessors ordered by $\prec$.

Linear and forward-branching models are the two most common types of instant-based models of time, where the former capture the view that time (or, the world) is deterministic, whereas the latter represent the view that only the past is deterministic and has no alternatives, where as the future is not deterministic but branches into many alternative possible futures. Both views are natural and meaningful, and each of them provides semantics for a family of temporal logics. These will be discussed respectively in Sections 4 and 5.

Another important distinction is between *discrete* and *dense*, or even *continuous* models of time. The former are typically used in artificial intelligence and in computer science, where the flow of time represents a discrete succession of events, transitions, or stages of a computational process. The latter usually represent 'real, physical time' and are more common in natural sciences.

There are examples of natural properties of instant-based models of time that cannot be expressed by first-order sentences, but require an essentially *second-order* logical language, with quantification not only over individual instants, but also over *sets of instants*. Some of the most important such examples are *Dedekind completeness*, *continuity*, *well-ordering*, *forward/backward induction*, and the *finite interval property*. I will informally describe these in linear models.

- **Dedekind completeness** means that every non-empty set of elements that is bounded above has a least upper bound. Examples are the ordered sets of the natural numbers, integers, and real numbers; while a non-example is the ordering of the rational numbers: for instance, the set of all rational numbers whose square is less than 2 is bounded above (say, by 2) but its least upper bound is $\sqrt{2}$, which is not a rational number.
- The property of **continuity** means that there are no 'gaps' in the precedence order. To be continuous, the temporal order must be both dense and Dedekind complete. Thus, the orderings of the natural numbers and of the integers are not continuous, but the ordering of the reals is.

- A linear instant-based model is **well ordered** if every non-empty set of instants has a least element. Equivalently, if there are no infinite (strictly) descending sequences of instants. Well-ordering is closely related (in a sense, even equivalent) to the **principle of transfinite induction** generalising the usual mathematical induction on natural numbers.
- A partial ordering is **forward inductive** if every infinite $\prec$-ascending sequence of instants is **co-final**, meaning that every instant precedes some instant in the sequence. In other words, such a sequence has no *strict upper bound*, and hence there are no 'transfinite instants' in the future. Respectively, it is **backward inductive**, if every infinite $\prec$-descending sequence of instants is **co-initial**, meaning that every instant succeeds some instant in the sequence, so such a sequence has no *strict lower bound*. Thus, every well-ordered model is vacuously backward inductive. A non-trivial example is the ordering of the integers, which is both backward and forward inductive, but it is not well ordered. No dense ordering is either backward or forward inductive; however, non-density is not sufficient to ensure either of these. For instance, extend the ordering of the natural numbers with an 'infinite number' ∞, greater than any of them. Then the sequence of all natural numbers is not co-final, as ∞ does not precede any of them, hence the resulting ordering is not forward inductive.
- Lastly, a linear model has the **finite interval property** if between any two elements there are only finitely many instants. This is the case precisely when it is both backward and forward inductive. Note that this property is incomparable with well-ordering. For example: the natural numbers are both well ordered and have the finite interval property; the integers (or, the negative integers) are not well ordered but still have the finite interval property; any transfinite ordinal (e.g., $\omega + 1$, the natural numbers extended with ∞), is well ordered but does not have the finite interval property; and the positive reals are neither well ordered nor do they have the finite interval property.

We will see in Section 3.6 that each of these properties can be expressed in a precise sense by means of propositional temporal formulae.

2.2 Interval-Based Models of Time

Instant-based models of time are often not suitable for reasoning about events with duration. To represent such events, one should rather use models with underlying temporal ontology based on *time intervals*, that is, time periods rather than time instants, as the primitive entities. The roots of interval-based temporal reasoning can be traced back to Zeno and Aristotle. Apparently, Zeno