1 Introduction

In recent decades network theory has been proven useful in describing many different complex systems from different disciplines, such as ecological systems (Paine, 1966; Pocock et al., 2012; Polis & Strong, 1996), epidemics (Pastor-Satorras et al., 2015; Wang et al., 2017), human microbiome (Gibson et al., 2016; Smillie et al., 2011), protein–protein interaction networks (Kovács et al., 2019; Li et al., 2017; Milo et al., 2002), finance (Stauffer & Sornette, 1999; Wei et al., 2014), climate (Donges et al., 2009; Fan et al., 2017; Ludescher et al., 2014), urban traffic (Hamedmoghadam et al., 2021; Li et al., 2015), and the human brain (Gallos et al., 2012; Moretti & Muñoz, 2013; Sporns, 2010). The resilience of complex networks is usually studied using percolation theory, which describes the robustness of the network under random failures or targeted attacks (Bunde & Havlin, 1991; Kirkpatrick, 1973; Stauffer & Aharony, 2018). Despite many advances in the field, especially in percolation of classical spatial structures such as lattices, most of the recent studies of more complex structures focus on nonspatial random networks, mainly because analytical solutions are easier to develop using different methods such as the generating functions formalism and mean-field approximations. However, when complex spatial networks are needed to describe real systems such as infrastructures (Latora & Marchiori, 2005; Yang et al., 2017), communication and transportation systems (Bell & Iida, 1997; Stork & Richards, 1992), or power grids (Yang et al., 2017), the preceding approaches fail and only limited analytical results for percolation theory are available.

In this Element, we will describe recent developments in the study of percolation in spatial networks. The Element is organized as follows: Section 2 will cover some basic principles of percolation theory for readers who are unfamiliar with this process. Section 3 will describe how the classical spatial structures of lattices were extended to more complex structures. We will also describe the structures of specific homogeneous and heterogeneous spatial networks and their behavior under the percolation process. In Section 4 we will describe the developments in recent years toward understanding the resilience of networks of networks and how they behave under percolation process in the presence of spatial constraints. Section 5 presents a new type of attack, localized attack on interdependent spatial networks, while Section 6 summarizes the conclusions.

2 Basic Remarks on Percolation

2.1 Classical Percolation

Percolation theory has been found useful to describe the structural phase transition of a network under random failures (Bunde & Havlin, 1991; Kirkpatrick, 1973; Stauffer & Aharony, 2018). In the percolation process, a fraction of $1 - p$ of nodes (site percolation) or edges (bond percolation) is randomly removed from the network, as shown in Fig. 1 for a 2D square lattice (of size $N = L \times L$). The functionality of the network is described by the relative size of the giant (largest) connected component (GCC), P_∞, and nodes that are not in the GCC are regarded as nonfunctional. Single networks experience a continuous second-order percolation phase transition where, above the critical percolation threshold, p_c, an infinite giant component (for an infinite

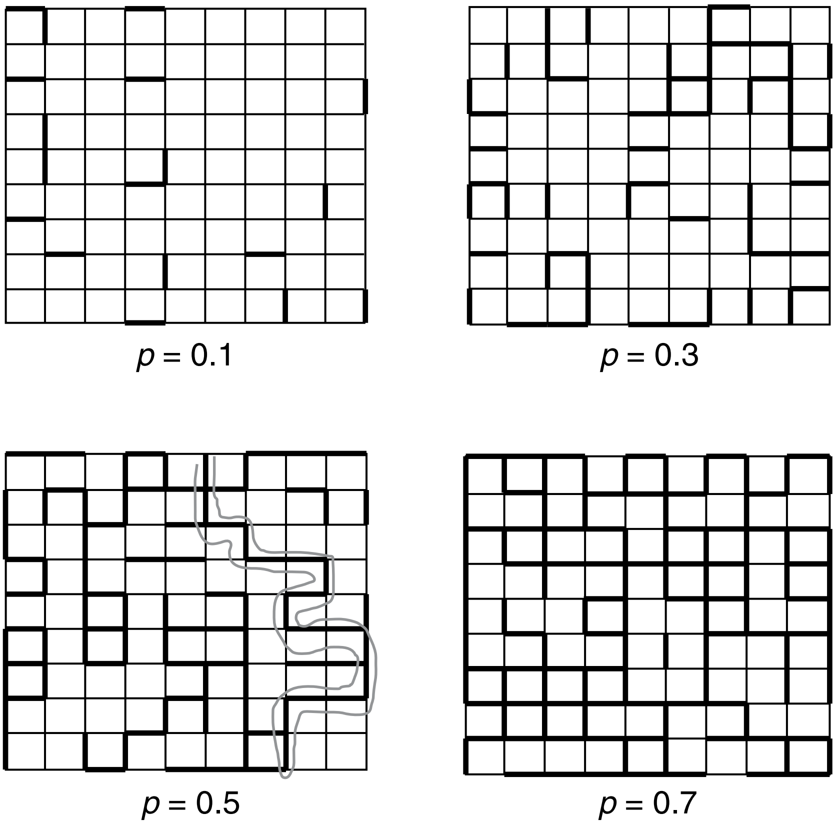

Figure 1 Demonstration of bond percolation on a 2D square lattice A fraction of $1 - p$ of edges is randomly removed from the network. At values lower than p_c ($p = 0.1$ and 0.3) only finite clusters exist and there is no path from one side of the network to the other (e.g., top to bottom). However, at $p = p_c = 1/2$ and higher ($p = 0.7$) a giant component that spans the entire network emerges, containing a path from one side of the network to the other. This phenomenon is known as a structural second-order phase transition, which usually characterizes percolation transition in a single network.
Source: Suki et al. 2011

network, $N \rightarrow \infty$) spanning the entire network exists, and the network is regarded as functional. Furthermore, in the case of spatial networks, the giant component contains a path from one side of the network to the other. In contrast, below the percolation threshold, the giant component is dismantled and only finite clusters remain (Fig. 1), and the network is regarded as nonfunctional. Except for the giant component, other important quantities can be evaluated during the process, such as the correlation length, ξ, which characterizes the typical length of finite clusters, and the susceptibility, χ, measured as the mean size of finite clusters.

Near (and at) the percolation threshold, a phase transition occurs and critical behavior is observed, represented by critical exponents:

$$P_\infty \sim (p - p_c)^\beta, \tag{2.1}$$

$$\xi \sim |p - p_c|^{-\nu}, \tag{2.2}$$

$$\chi \sim |p - p_c|^{-\gamma}. \tag{2.3}$$

In addition, several dimensional quantities exist that define different properties of the network. The first is the network dimension d relating the number of nodes in the network N to the linear size of the network L as $N = L^d$. In the case of random networks where L is not defined, the network is characterized by the small-world property $l \sim \log N$ (Milgram, 1967) (l is the chemical length, also called the minimal number of hops between two random sites), and the exponential increase of N with l indicates infinite dimension. Since the dimension of a random network is always larger than any network embedded in any d (N of random networks is larger with every L^d due to its exponential increase with l), it is impossible to embed random networks in space. Second is the fractal dimension $d_f < d$ of the giant component at the percolation threshold, $S = P_\infty \times N \sim L^{d_f}$. And finally, there is the dimension of the shortest path on the giant component d_{min}, which is defined as (Havlin & Nossal, 1984)

$$l \sim \langle r \rangle^{d_{min}}, \tag{2.4}$$

where l is the shortest path (called also chemical length, as mentioned), and $\langle r \rangle$ is the mean Euclidean distance between two sites at shortest path l. These dimensions together with the critical exponents satisfy a variety of scaling relations, such as $\beta + \gamma = \nu d_f$ and $d_f = d - \beta/\nu$ (Stanley, 1971), indicating that the critical exponents are not independent, and we can find all critical exponents by knowing only two of them.

The percolation phase transition and its critical exponents are significantly affected by the network structure. Interestingly, the concept of *universality class* groups together different systems with similar properties. For example,

Table 1 Some percolation-critical exponents. The full list can be found in Bunde and Havlin (1991).

d	2	6+
β	5/36	1
ν	4/3	1/2
γ	43/18	1
δ	91/5	2
d_f	91/48	4
d_{min}	1.13	2

the 2D square lattice and the 2D hexagonal lattice belong to the same universality class and share the same critical exponents (although with different p_c). Moreover, for all types of lattices an upper critical dimension $d_c = 6$ exists. Networks with dimension 6 or larger belong to the same university class, which is sometimes called the *mean-field* university class. Thus, while random networks do not follow the preceding definitions of d, d_f, and d_{min} since L and $\langle r \rangle$ are not defined, they take the values of a 6-dimensional lattice for all relevant scaling relations.

Navigation in Networks

As a side remark, let us mention here the problem of *navigation* in networks, which shares some features of percolation. The problem of navigation in networks deals with the transmission of information between different nodes in the system (Milgram, 1967). In the basic model, at each step the information is transmitted to the closest node to the target. If the entire network structure is known at each step, the navigation route will simply be the shortest path, described by d_{min}. However, if only *local* properties are known, interesting routes characterized by different properties emerge (Kleinberg, 2000; Viswanathan et al., 1999). Moreover, additional interesting features appear when the networks are embedded in space (Hu et al., 2011; Huang et al., 2014), and the interested reader may study this topic as well.

Some of the most studied structures under the percolation process are random graphs. Random graphs with a Poisson degree distribution, $P(k) = \frac{z^k e^{-z}}{k!}$, where k is the degree (i.e., the number of links per node, with z being the average degree) are known as Erdős–Rényi (ER) networks (Erdős & Rényi, 1959; Erdős & Rényi, 1960). While space is irrelevant in random graphs, it was found that geometric metrics can provide important insight into the percolation transition (Boguñá et al., 2021). Percolation phase transition on ER networks is the same as for lattices of dimension equal or larger than 6 and also belongs to the mean-field universality class. The percolation properties of ER networks can be analytically solved using the generating functions formalism (Newman et al., 2001), and the size of the giant component can be obtained from the transcendental equation (Bollobás, 1985; Erdős & Rényi, 1959; Erdős & Rényi, 1960)

$$P_\infty = p(1 - e^{-zP_\infty}), \tag{2.5}$$

with $p_c = 1/z$.

A similar well-studied random graph structure is the random regular (RR) network. In RR networks all the nodes have the same degree k_0 and the degree distribution is $P(k) = \delta_{k,k_0}$. Since the neighborhood of all nodes in ER and RR is the same and deviations from average are small, these networks fulfill translational symmetry and therefore belong to the same universality class and share the same critical exponents.

Another important random graph structure is the scale-free (SF) network. This structure describes many real networks such as the World Wide Web (Barabási & Albert, 1999), the Internet (Faloutsos et al., 1999), biological networks (Jeong et al., 2000), and airline networks (Colizza et al., 2006). Scale-free networks are characterized by a power-law degree distribution,

$$P(k) \sim k^{-\lambda}. \tag{2.6}$$

In contrast to ER and RR networks, the existence of hubs in SF breaks the translational symmetry (in which every node sees a similar neighborhood), leading to a different universality class with different critical exponents. Scale-free networks with $\lambda \leq 3$ have been proven to be very robust with $p_c = 0$ due to the existence of hubs (high-degree nodes), which are rare but connect the network even after many random failures (Cohen et al., 2000).

While the generating functions approach describes random graphs well, it fails to describe spatial networks due to its local treelike structure assumption, which is usually not valid in spatial networks (Barthélemy, 2011; Gastner & Newman, 2006). Thus, analytical results for percolation in spatial networks are limited, with only a few results for the critical exponents in 2D (Den Nijs, 1979; Nienhuis, 1982) and a few analytical results such as $p_c = 1/2$ for bond percolation on a 2D square lattice (Fig. 1) (Sykes & Essam, 1964).

2.2 Network Resilience

While the classical percolation process is used to study network resilience under random failure, in recent decades percolation has been generalized for studying network resilience under other failure processes like targeted attack and localized attack. In a targeted attack, nodes are not randomly removed and instead central nodes or high-degree nodes are removed first. Albert et al. (2000) showed that in ER networks the percolation threshold is similar for random failures and targeted attacks. In contrast, scale-free networks are much more vulnerable to targeted attacks due to the existence of hubs. It was proven analytically by Cohen et al. (2000) that for random failures on SF networks with $\lambda \leq 3$, $p_c = 0$, while for targeted attack p_c is close to 1 (Cohen et al., 2001). That is, for random failures, unless one removes all nodes, the SF network does not collapse; however, for targeted attacks on high-degree nodes only a small fraction needs to be removed and the network collapses. One can take advantage of this property in order to efficiently immunize a population (or a computer network) by finding and immunizing the high-degree nodes (hubs) or super-spreaders in the population, which then block the spreading channels of the disease (Cohen et al., 2003; Liu et al., 2021). However, note that the case of targeted attacks on a spatial structure such as a 2D square lattice is trivial since the removal of a strip breaks the system into two clusters, which in the thermodynamic limit results in $p_c = 1$.

Localized Attack Localized attacks in networks, in contrast to random failures, deal with the case where damage is not randomly spread in the network but rather localized in a given region of the network or a neighborhood of a given node. A simple scenario of localized attack in spatial structures is the creation of a hole of a given radius size r at the center of the network. Here also, the case of a spatial structure such as a 2D square lattice is trivial, and the critical radius size, r_c, where the system collapses is simply of the order of the system size, L, which in the thermodynamic limit $r_c \to \infty$ and corresponds to $p_c = 0$. Localized attacks have also been studied in random structures using the shell structure. Instead of creating a hole with geometric radius r, which has no meaning without space, a hole with shell radius l is removed. This means that a single node is removed, which corresponds to $l = 0$; then its neighbors are removed, which corresponds to $l = 1$, and so on until a fraction of $1-p$ of nodes is removed. Shao et al. (2015) studied localized attacks on random graphs and found that the percolation threshold for random attack and localized attack on ER networks is the same. In contrast, SF networks are characterized by different behavior depending on the power-law exponent λ. For $\lambda < \lambda_c = 3.825$, the percolation threshold of localized attack is larger compared to that of random attack (i.e., localized attacks cause more damage compared to random failures), and for $\lambda > \lambda_c$ it is the opposite. This is since for small λ it is easy to find the hubs in the neighborhood of a random node, and localized attacks become more efficient in breaking the network compared to random attacks.

3 Beyond Classical Structures

The classical structure of spatial networks involves nearest neighbors connections with a fixed degree of each node, such as a 2D square lattice or a 2D hexagonal lattice or higher-dimensional lattices. These models have been extended to the case where the degree of nodes follows a distribution such as scale-free with $P(k) \sim k^{-\lambda}$ and the links were extended, if needed, to second and third nearest neighbors, and so on. In such cases the dimension of the shortest path (Eq. (2.4)) has been shown to be

$$d_{min} = (\lambda - 2)/(\lambda - 1 - 1/d). \tag{3.1}$$

Thus, for $d > 1$, due to the long-range links, the dimension $d_{min} < 1$. This result is the opposite of any lattice structure, which is known to have $d_{min} > 1$, and shows how even a small spatial change in the network structure can significantly change the properties of the network.

These studies (Ben-Avraham et al., 2003; Rozenfeld et al., 2002) broke free from the stiff shell of the classical nearest neighbors structures and opened a new direction for spatial embedded networks.

3.1 Homogeneous Structures

Studies of real-world spatial embedded networks show that, in many cases, the nearest neighbors linking structure, like lattices, is not realistic and that the Euclidean length of the networks' edges, r, follows a certain length distribution, $P(r)$. Bianconi et al. (2009) show that the distribution of edge lengths of the global airline network follows $P(r) \sim r^{-3}$. Lambiotte et al. (2008) show that the mobile phone communication network follows $P(r) \sim r^{-2}$ (see also Goldenberg and Levy (2009)), and Liben-Nowell et al. (2005) show that the spatial distribution of distances between social network friends follows $P(r) \sim r^{-1}$. To generally describe such systems, Daqing et al. (2011) developed a model of a random graph embedded in space where the nodes are the lattice sites with a Poisson degree distribution, $P(k) = \frac{z^k e^{-z}}{k!}$, and a power-law distribution of edge lengths,

$$P(r) \sim r^{-\delta}. \tag{3.2}$$

This model recovers the nearest neighbors structures of lattices for $\delta \to \infty$ and becomes a random graph for $\delta = 0$; see Fig. 2. Li et al. (2011) followed by Emmerich et al. (2013) studied this model and found that when the network is embedded in 2D, both its dimension and percolation properties change according to δ. For $\delta > 4$, there are very few long-range connections, and the network belongs to the universality class of percolation in a 2D lattice. For $2 < \delta < 4$, there are more long-range connections, and the percolation properties show new intermediate behavior different from mean-field, the dimension of the network increases continuously from $d = 2$ with decreasing δ, and the critical exponents depend on δ. Finally, for $\delta \leq 2$, there are many long-range connections, the dimension d becomes infinite, and the percolation transition belongs to percolation of random graphs. When the network was embedded in 1D, it was found that for $\delta \leq 1$, the dimension is infinite and the percolation transition is mean-field. For $1 < \delta < 2$, the dimension decreases and the critical

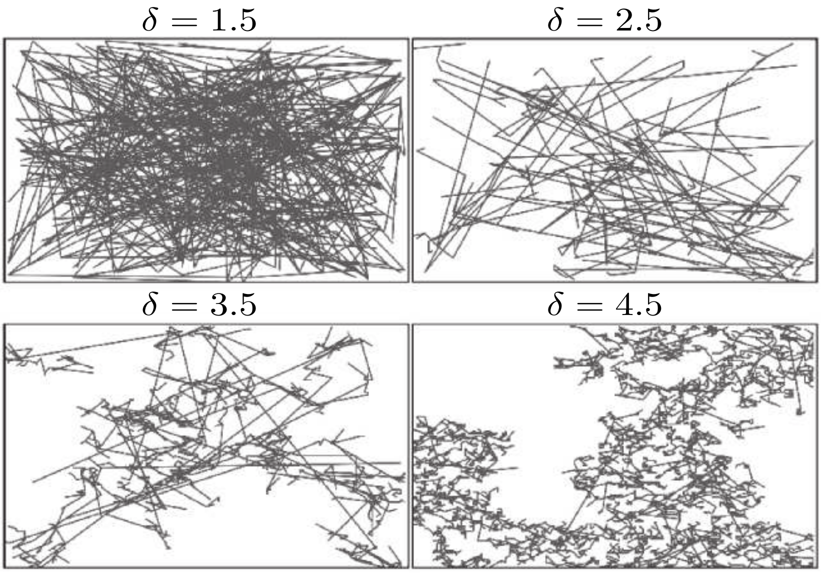

Figure 2 Demonstration of ER network embedded in space with power-law distribution of edge-length, $P(r) \sim r^{-\delta}$ The giant component of these networks embedded in a 2D square lattice at the percolation threshold with $\delta = 1.5, 2.5, 3.5$, and 4.5 is shown. Since the edges are distributed randomly between nodes, the degree distribution is Poissonian, $P(k) = \frac{z^k e^{-z}}{k!}$ as in an ER network. As δ increases, the giant component has fewer long-range connections and becomes more affected by the constraints of the embedding space, as in 2D lattices with fractal dimension $d_f = 91/48$ (Bunde & Havlin, 1991; Stauffer & Aharony, 2018). **Source:** Li et al. 2011.

exponents depend on δ, and for $\delta > 2$, there is no percolation transition as in a 1D lattice. Schmeltzer et al. (2014) also studied this random graph embedded in space model, but with degree correlations. They showed that the percolation threshold can change in different ways according to the mechanism at which the degree correlation is induced in the model.

The ζ-Model In contrast to our preceding examples of real-world networks that we have discussed so far, which are characterized by a power-law distribution of edge lengths, other real-world networks are characterized by an exponential distribution of edge lengths. Examples of such networks include transport systems or power grids (Danziger et al., 2016; Halu et al., 2014) and brain networks (Bullmore & Sporns, 2012; Ercsey-Ravasz et al., 2013; Horvát et al., 2016; Markov et al., 2014). The first model describing such systems is the Waxman model (Waxman, 1988). In the Waxman model, the distribution of edge length follows an exponential distribution,

$$P(r) \sim \exp(-r/\zeta). \tag{3.3}$$

Here, ζ is the characteristic length of the links, and the nodes are randomly distributed in space. Inspired by the Waxman model, a similar model with the

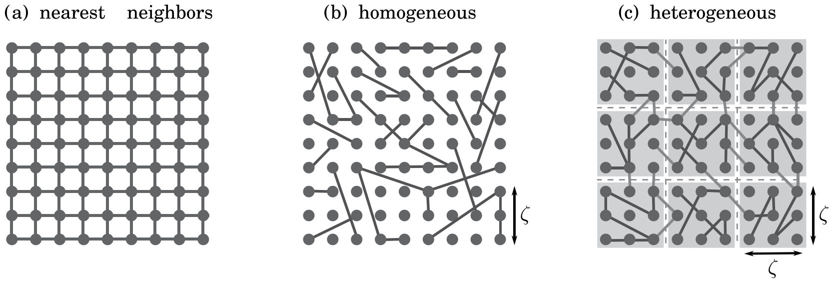

Figure 3 2D spatial embedded network models (a) The basic spatial models are the classical nearest neighbors models such as a 2D square lattice. **(b)** Spatial homogeneous models are extensions of the classical nearest neighbors models (in **(a)**), where the links are homogeneously distributed according to link-length distribution, $P(r)$. The example here demonstrates the ζ-model with the link-length distribution described in Eq. (3.3). **(c)** Demonstration of a spatial heterogeneous model which consists of communities of linear size ζ and distinct internal organization (brown edges). The intra-links can connect any pair of nodes within the communities. The communities are placed next to each other in a given lattice of dimension d, and only nearest-neighbor communities can be interconnected. The spatial effect is induced by the interconnections between neighboring communities in space (pink edges).

same edge-length distribution, Eq. (3.3), was developed (Danziger et al., 2016) where the nodes are not randomly distributed in space but rather organized as the sites in a 2D square lattice (see Fig. 3b) or other dimensional lattices. We call this model the *ζ-model* due to the spatial control parameter of the characteristic length ζ. Similar to the network having the distribution described in Eq. (3.2), the ζ-model having the distribution of Eq. (3.3) also recovers the nearest neighbors lattice structures in the limit of $\zeta \to 0$ and random graph structure for $\zeta \to \infty$. However, the main difference between both models, represented by Eq. (3.2) and Eq. (3.3), is the absence of long-range connections and having a characteristic length scale ζ in the ζ-model.

The ζ-model has been extensively studied in recent years under the percolation process (Bonamassa et al., 2019; Danziger et al., 2016, 2020; Gross et al., 2017). It was found that the percolation threshold of the two extreme limits $\zeta \to 0$ and $\zeta \to \infty$ is at the lattice threshold $p_c \simeq 0.5926$ and the random graph threshold $p_c = 1/z$ respectively, while the percolation threshold of intermediate values of ζ are between these limits (see Fig. 4a).

The varying length scale ζ in the ζ-model has been found to create a unique phenomenon of stretching which has not been observed before in the classical lattice model or in the power-law length distribution model (Eq. (3.2)). In short scales below ζ, each pair of nodes is likely to be connected with the same probability in a manner similar to that of a random graph with infinite dimension, while on scales longer than ζ a spatial behavior of $d = 2$ is observed due to the absence of long-range connections. This structural property leads the system to have a *crossover* between random behavior with infinite dimension in short scales (below ζ) and spatial behavior with $d = 2$ for long scales (see Fig. 4b). A similar crossover is observed at the percolation threshold; see Fig. 4c. Since the dimension of the giant component at the percolation threshold is fractal (Bunde & Havlin, 1991), a crossover in the fractal dimension is also seen. That is, at criticality, a mean-field fractal dimension $d_f^{MF} = 4$ is observed in short scales, and a 2D fractal dimension $d_f^{2D} = 91/48$ is observed in long scales (Fig. 4c).

This crossover phenomenon is also observed in the critical exponents close to the percolation threshold. An example of such a crossover is observed in the correlation length which is characterized by $\nu_{2D} = 4/3$ for 2D spatial structure and $\nu_{MF} = 1/2$ for random graphs and can now be described as (see Fig. 4d)

$$\xi \sim |p - p_c|^{-\nu} \text{ where } \begin{cases} \nu = 4/3, & \text{for} \quad p^* < p < p_c \\ \nu = 1/2, & \text{for} \quad p < p^* \end{cases} . \tag{3.4}$$