1

Human

Problem 1 Human Eye

How do we see?
What kind of glasses might we need?
When can we distinguish between the two eyes of a cat during the night?

A schematic view of the structure of the human eye is presented in Figure 1.1. Light rays that refract at the cornea and eye lens end up at the retina, which produces nerve impulses sent to the brain down the optic nerve. In a simplified model of an eye, the cornea and eye lens can be replaced with one converging lens (called simply the lens in the remainder of the text) while the retina can be modeled as a disk of radius $R = 1.00\,\mathrm{cm}$, the axis of which coincides with the optical axis of

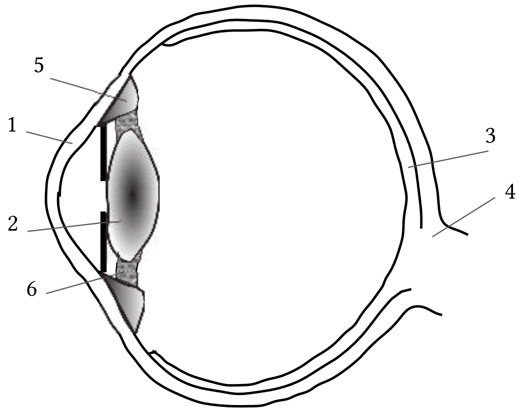

Figure 1.1 Scheme of the structure of the human eye: (1) cornea, (2) eye lens, (3) retina, (4) optic nerve, (5) ciliary muscles, (6) suspensory ligament

the lens, as shown in Figure 1.2. The distance between the retina and the lens is $d = 2.40\,\text{cm}$. A human can adjust the focal length of the lens and therefore has the capability of clearly seeing objects at different distances. This process is called eye accommodation and is enabled by ciliary muscles connected to the eye lens by a suspensory ligament. These muscles act to tighten or relax the ligaments and therefore thin down or thicken the lens. Consequently the focal length of the lens changes.

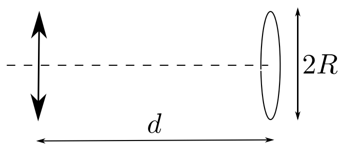

Figure 1.2 A simplified model of the human eye

(a) A human has regular eyesight if images of all objects from a distance larger than $d_0 = 25.0\,\text{cm}$ can be formed at the retina. What is the range of the lens' focal lengths for a human with regular eyesight?
(b) The maximal focal length f_{max} of the lens for a nearsighted man is smaller than the upper limit of the range determined in part (a). This man uses glasses with a diopter value of $D_1 = -1.00\,\text{m}^{-1}$ to clearly see very distant objects. Determine f_{max} and find the maximal distance of an object that this man can clearly see without using the glasses. For simplicity neglect the distance between the glasses and the lenses.
(c) The minimal focal length f_{min} of the lens for a farsighted woman is larger than the lower limit of the range determined in part (a). This woman needs glasses with a diopter value of $D_2 = 2.00\,\text{m}^{-1}$ to clearly see objects at a distance of $d_0 = 25.0\,\text{cm}$. Determine f_{min} and find the minimal distance of an object that this woman can clearly see without using the glasses.
(d) A person is nearsighted (farsighted) as well when the distance between the retina and the lens is larger (smaller) than the regular distance of $d = 2.40\,\text{cm}$. Calculate the diopter value of the glasses that should be used by a man with a distance between the retina and the lens of $d_1 = 2.50\,\text{cm}$ ($d_2 = 2.30\,\text{cm}$).

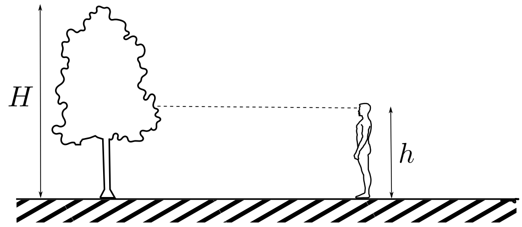

Figure 1.3 With problem 1(e)

(e) A man with regular eyesight whose height is $h = 2.00\,\text{m}$ is observing a tree of height $H = 2h$ (Figure 1.3). His view is directed toward the middle of the tree. What is the minimal distance between the man and the tree that allows him to see the whole tree?

Two types of light receptors are placed at the retina – rods (about $N_1 = 10^8$ of them) and cones (about $N_2 = 6 \cdot 10^6$ of them). Rods enable night vision, while cones are used for vision during the day. Assume that a person can distinguish two distant objects during the day (night) if their images are at different cones (rods). Assume also that the cones (rods) are evenly distributed on the retina surface and that their positions form a square lattice.

(f) Two point objects are at a mutual distance of $a = 1.00\,\text{mm}$. The direction that connects them is perpendicular to the optical axis of the lens (Figure 1.4). What is the maximal distance from which a woman can distinguish between these two objects during the day?

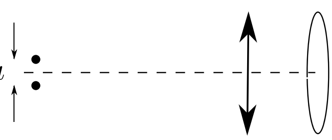

Figure 1.4 With problem 1(f)

(g) At what maximal distance can a woman read the license plates of a car during the day? Assume that the license plates can be read if a woman can distinguish between the point objects at a mutual distance of $a = 1.00\,\text{cm}$.

(h) At what maximal distance can a woman distinguish between the two eyes of a cat during the night? The eyes of a cat are at a mutual distance of $a = 2.00\,\text{cm}$.

Solution of Problem 1

(a) To see an object at a distance p from the eye, a human needs to accommodate the focal length of the lens so that the image of the object is formed at the retina (which is at a distance $l = d$ from the lens). For an object at a distance $p_1 = d_0$ the focal length is given by lens equation $\frac{1}{f_1} = \frac{1}{p_1} + \frac{1}{l}$. For an object at a distance $p_2 \to \infty$ we have $\frac{1}{f_2} = \frac{1}{p_2} + \frac{1}{l}$. From previous equations we obtain $f_1 = 2.19\,\text{cm}$ and $f_2 = 2.40\,\text{cm}$. Consequently the lens focal length of a human with regular eyesight takes a range from $f_1 = 2.19\,\text{cm}$ to $f_2 = 2.40\,\text{cm}$.

(b) The lens focal length and the distance of the object that the man clearly sees are related by $\frac{1}{f} = \frac{1}{p} + \frac{1}{d}$. Consequently, without the use of glasses, this man cannot clearly see objects at a distance larger than $p_{\max}$, where

$$\frac{1}{f_{\max}} = \frac{1}{p_{\max}} + \frac{1}{d}. \tag{1.1}$$

The focal length of the system lenses-glasses $f_{\rm ns}$ satisfies the relation $\frac{1}{f_{\rm ns}} = \frac{1}{f} + D_1$. When this man clearly sees very distant objects with the use of glasses, the lens equation reads

$$\frac{1}{f_{\max}} + D_1 = \frac{1}{p_2} + \frac{1}{d}, \tag{1.2}$$

where $p_2 \to \infty$. From equation (1.2) we obtain $f_{\max} = \frac{d}{1-dD_1} = 2.34\,\text{cm}$. By subtracting equations (1.1) and (1.2) we find $p_{\max} = -\frac{1}{D_1} = 1.00\,\text{m}$.

(c) Without the use of glasses, this woman cannot clearly see objects at a distance smaller than $p_{\min}$, where

$$\frac{1}{f_{\min}} = \frac{1}{p_{\min}} + \frac{1}{d}. \tag{1.3}$$

The lens equation for a woman with glasses looking at an object at a distance d_0 reads

$$\frac{1}{f_{\min}} + D_2 = \frac{1}{d_0} + \frac{1}{d}. \tag{1.4}$$

From equation (1.4) it follows that

$$f_{\min} = \frac{1}{\frac{1}{d_0} + \frac{1}{d} - D_2} = 2.29\,\text{cm}. \tag{1.5}$$

By subtracting equations (1.3) and (1.4) we obtain

$$p_{\min} = \frac{d_0}{1 - D_2 d_0} = 50.0\,\text{cm}. \tag{1.6}$$

(d) The lens equation for a man with regular distance between the lens and the retina when he clearly sees an object at a distance p is $\frac{1}{f} = \frac{1}{p} + \frac{1}{d}$. For a man with distance d_i between the retina and the lens who uses glasses with diopter value D_i and clearly sees the same object when the lens focal length is the same, we obtain $\frac{1}{f} + D_i = \frac{1}{p} + \frac{1}{d_i}$. Subtracting the previous two equations, we find $D_i = \frac{1}{d_i} - \frac{1}{d}$. Consequently, we find in the first case $D_1 = -1.67\,\text{m}^{-1}$ and in the second case $D_2 = 1.81\,\text{m}^{-1}$.

(e) A man sees the whole tree when the size L of the image of the tree on the retina is smaller than the retina diameter (Figure 1.5). Using the similarity of the triangles in Figure 1.5 we obtain $\frac{L}{H} = \frac{d}{x}$, where x is the distance between the man and the tree. Consequently the man sees the whole tree when $L = H\frac{d}{x} < 2R$, leading to $x > \frac{Hd}{2R} = 4.80\,\text{m}$.

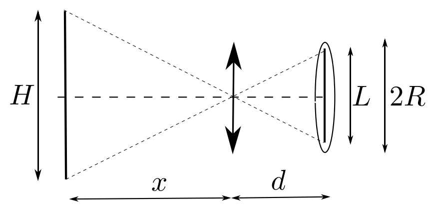

Figure 1.5 With the solution of problem 1(e)

(f) The number of cones per unit surface is equal to $N_S = \frac{N_2}{R^2\pi}$. On the other hand, since we assume that the positions of cones form a square lattice with lattice constant b, we also have $N_S = \frac{1}{b^2}$. From the previous two equations it follows that $b = R\sqrt{\frac{\pi}{N_2}} = 7.24\,\mu$m. When the woman is at a maximal distance at which she can still distinguish between the two objects, the images of the objects are formed at two neighboring cones. From the similarity of triangles in Figure 1.6, we find $\frac{a}{x} = \frac{b}{d}$ – that is, $x = \frac{ad}{b} = 3.32$ m.

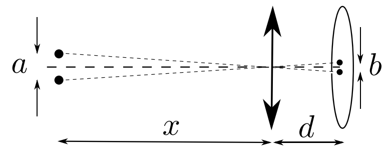

Figure 1.6 With the solution of problem 1(f)

(g) From the solution of part (f) we have $x = \frac{ad}{b}$, where in this case $a = 1.00$ cm, leading to $x = 33.2$ m.

(h) Since the woman observes the cat during the night, the solution of part (f) is modified only by replacing the number of cones with the number of rods. Consequently, $x = \frac{ad\sqrt{N_1}}{R\sqrt{\pi}} = 271$ m.

We refer the reader interested in more details regarding the physics of the human eye to chapter 12, reference [13].

Problem 2 The Circulation of Blood

How powerful is the human heart?
How does a bypass help in the case of arteriosclerosis?

The human cardiovascular system consists of the heart, the blood, and the blood vessels. The heart pumps the blood through the blood vessels. The blood carries nutrients and oxygen to and carbon dioxide away from various organs. The most important portions of the cardiovascular system are pulmonary circulation and systemic circulation. Pulmonary circulation pumps away oxygen-depleted blood from the heart via the pulmonary artery to the lungs. It then returns oxygenated blood to the heart via the pulmonary vein. Systemic circulation transports oxygenated blood away from the heart through the aorta. The aorta branches to arteries that bring the blood to the head, the body, and the extremities. The veins then return oxygen-depleted blood to the heart. The direction of blood flow is determined by four heart valves. Two of them are positioned between the antechambers and the chambers, while two are located between the chambers and the arteries.

(a) The heart pumps blood by contraction of the muscles of the antechambers and chambers. The blood pressure gradually increases from the minimal (diastolic) value of $p_d = 80\,\text{mmHg}$ to the maximal (systolic) value of $p_s = 120\,\text{mmHg}$ during contraction.

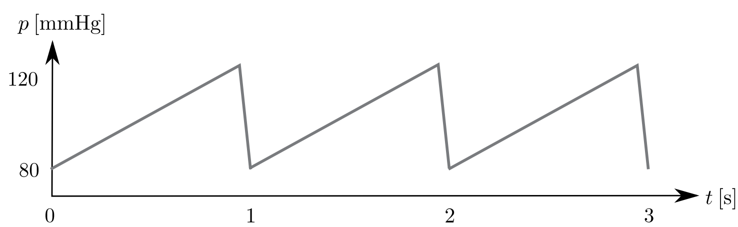

Figure 1.7 The graph of the dependence $p(t)$

The muscle then relaxes and the value of pressure suddenly decreases, as shown in Figure 1.7. The heart contracts (beats) around 60 times a minute. Each contraction pumps around 75 ml of blood. The pump shown in Figure 1.8 is a simple model of the heart. The heart decreases the volume during the contraction, which corresponds to the upward motion of the piston in the model.

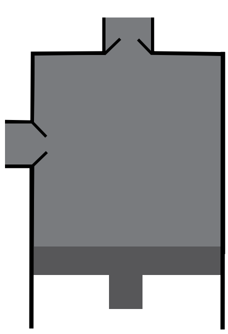

Figure 1.8 A pump as a model of the heart

Thereby the pressure increases and closes the input valves while it opens the output valves. Determine the power of the heart.

The boundary between laminar and turbulent flow of blood is determined from the Reynolds number, which is directly proportional to the speed of blood v. The Reynolds number is a dimensionless quantity that depends as well on the density of blood $\rho = 1,060\,\mathrm{kg/m^3}$, viscosity of blood $\eta = 4.0 \cdot 10^{-3}\,\mathrm{Pa \cdot s}$ and the diameter of the blood vessel D. The flow is turbulent if the Reynolds number is larger than 2,000, while it is laminar otherwise.

(b) Derive the expression for the Reynolds number using dimensional analysis. Assume that the dimensionless constant that appears in front of the expression is equal to 1.
(c) The diameter of the aorta is $D = 10\,\mathrm{mm}$. Calculate the maximal speed of laminar blood flow in the aorta.

We consider next the laminar flow of blood through the artery whose shape is a cylinder of length L and radius R, as shown in Figure 1.9. The flow of blood in the artery is caused by the difference of pressures Δp at the ends of the artery, which is a consequence of blood pumping from the heart. The blood does not slide at the walls of the artery. For this reason, a cylindrical layer of blood that is at rest is formed near the wall of the artery. The viscosity of the blood causes laminar flow where each layer slides between neighboring layers. The viscosity force between the layers F is given by Newton's law,

$$F = \eta S \frac{\Delta v}{\Delta r},$$

where η is the viscosity of the blood, S is the area of the layer that is in contact with the neighboring layer, and $\Delta v / \Delta r$ is the gradient of speed in the radial direction. The walls of the artery are inelastic and the speed of flow does not change between the points on the same line in the direction of the artery.

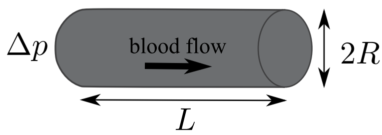

Figure 1.9 Artery

(d) Determine the dependence of the speed of blood on the distance from the artery axis.
(e) Using the analogy of electrical resistance, one can define the resistance of blood flow as the ratio of the pressure difference and the volume flow caused by this difference of pressures. Determine the blood flow resistance through the artery.

(f) As a consequence of arteriosclerosis, the inner diameter of a part of the artery decreased from $d_1 = 6.0\,\text{mm}$ to $d_2 = 4.0\,\text{mm}$. How many times was the blood flow resistance increased in this part of the artery? To reduce the blood flow resistance, a bypass can be introduced. A healthy artery or vein is removed from another part of the patient's body and attached in parallel to this part of the artery. Assume that the bypass is of the same length as this part of the artery. How many times does the blood flow resistance decrease after the introduction of a bypass of diameter $d_3 = 5.0\,\text{mm}$?

When the blood enters the artery, the speed of the blood is nearly the same throughout the cross-section of the artery. This means that the blood needs to accelerate and decelerate to reach the regime considered in previous parts of the problem. The blood near the artery walls decelerates to zero speed, while the part in the center of the artery accelerates to the maximal value of the speed. Consider the situation when we neglect the viscosity and when the blood accelerates along the artery.

(g) Determine the relation between the pressure difference Δp at the ends of the artery and the change of volume flow $\Delta q/\Delta t$ as a function of blood density ρ, the length of the artery L, and its radius R.

(h) As in part (e), the analogy with electrical circuits can be also introduced in part (g). Which element of the electric circuit can be used to describe the relation determined in part (g)?

Solution of Problem 2

(a) The work performed by the pump when the piston moves by Δr is

$$\Delta A = F\Delta r = \frac{F}{S}\Delta r S = p\Delta V\,. \tag{1.7}$$

The work performed by the heart is equal to the area under the graph of the function $p(V)$. The heart performs 60 beats per minute, which is 1 beat per second. Consequently the heart pumps in $V = 75\,\text{ml}$ of blood each second. Therefore, the graph of the function $p(V)$ looks as shown in Figure 1.10. The work performed by the heart during 1 beat is

$$A = p_d V + \frac{1}{2}(p_s - p_d)V = \frac{1}{2}(p_s + p_d)V = 1.0\,\text{J}\,. \tag{1.8}$$

The work A is performed by the heart during $t = 1\,\text{s}$, which means that the corresponding power is $P = A/t = 1\,\text{J}/1\,\text{s} = 1.0\,\text{W}$.

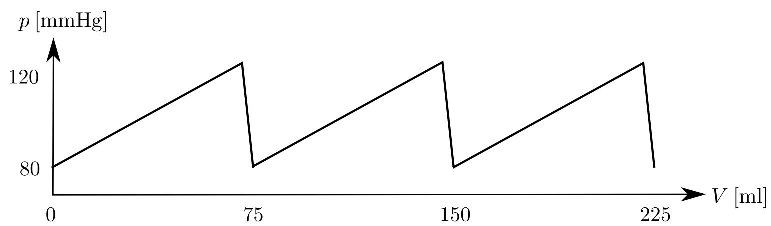

Figure 1.10 The graph of the function $p(V)$

(b) We can find the expression for the Reynolds number using dimensional analysis

$$\mathrm{Re} = v\rho^{\alpha}\eta^{\beta}D^{\gamma} \Rightarrow 1 = [\mathrm{m}\cdot\mathrm{s}^{-1}]\,[\mathrm{kg}\cdot\mathrm{m}^{-3}]^{\alpha}\,[\mathrm{kg}\cdot\mathrm{m}^{-1}\mathrm{s}^{-1}]^{\beta}\,[\mathrm{m}]^{\gamma}, \tag{1.9}$$

which leads to the system of equations

$$1-3\alpha-\beta+\gamma=0, -1-\beta=0, \alpha+\beta=0, \tag{1.10}$$

whose solution is $(\alpha,\beta,\gamma) = (1,-1,1)$. Therefore, the Reynolds number is given by the expression

$$\mathrm{Re} = \frac{\rho v D}{\eta}. \tag{1.11}$$

(c) The maximal speed of blood in the aorta is obtained for $\mathrm{Re} = 2,000$ and reads

$$v = \frac{\eta \mathrm{Re}}{\rho D} = 75\,\frac{\mathrm{cm}}{\mathrm{s}}. \tag{1.12}$$

The Reynolds number reaches the critical value when the valves of the aorta open. The blood is then under big pressure and reaches a speed as high as 120 cm/s. So-called Korotkoff sounds appear then as a consequence of turbulent flow. These can be heard using a stethoscope. This fact is used when blood pressure is measured using a sphygmomanometer.

(d) The system has cylindrical symmetry. Consequently the speed of blood is constant in each thin cylindrical layer. Consider the part of blood in the shape of a cylinder of radius r. This part of blood in the artery moves due to pressure difference Δp, which yields the force $F_1 = \pi r^2 \Delta p$. The magnitude of the viscosity force that acts on this layer is $F_2 = 2\pi r L \eta \frac{\Delta v}{\Delta r}$. Since each layer of blood is moving at a constant velocity, we obtain from Newton's first law that

$$F_1 = F_2 \Rightarrow \Delta v = \frac{\Delta p}{2\eta L} r \Delta r. \tag{1.13}$$

By transforming the equation (1.13) to differential form and performing integration with the boundary condition $v(R) = 0$, we obtain the dependence of the speed of blood on the distance from the axis of the artery:

$$v(r) = \frac{\Delta p}{4\eta L}(R^2 - r^2)\,. \tag{1.14}$$

(e) The flow of blood through the ring of width dr, which is located in the region between r and $r+dr$, is $v(r)dS$, where $dS = 2\pi r dr$ is the area of that ring. The flow of blood through the artery is then obtained by performing the integration over all rings, which leads to

$$q = \int_0^R v(r) 2\pi r dr = \int_0^R \frac{\pi \Delta p}{2\eta L}(rR^2 - r^3)dr = \frac{\pi \Delta p R^4}{8\eta L}\,, \tag{1.15}$$

and consequently the blood flow resistance is

$$\mathcal{R} = \frac{\Delta p}{q} = \frac{8\eta L}{\pi R^4}\,. \tag{1.16}$$

(f) Due to arteriosclerosis the blood flow resistance in the sick part of the artery $\mathcal{R}_2$ increases in comparison to the resistance in the healthy artery $\mathcal{R}_1$, which leads to

$$\frac{\mathcal{R}_2}{\mathcal{R}_1} = \left(\frac{d_1}{d_2}\right)^4 = 5.1\,. \tag{1.17}$$

After the bypass is introduced, the sick part of the artery and the bypass form a parallel connection of two resistors with equivalent resistance $\mathcal{R}_e$. The resistance then reduces by

$$\frac{\mathcal{R}_2}{\mathcal{R}_e} = \frac{\mathcal{R}_2}{\frac{\mathcal{R}_2\mathcal{R}_3}{\mathcal{R}_2+\mathcal{R}_3}} = 1 + \left(\frac{d_3}{d_2}\right)^4 = 3.4\,. \tag{1.18}$$

(g) Newton's second law applied to the blood in the artery gives:

$$m\frac{\Delta v}{\Delta t} = \Delta p S\,, \tag{1.19}$$

where $m = \rho V = \rho L \pi R^2$ is the mass of the blood in the artery, $\Delta v/\Delta t$ is the change of the speed of blood along the artery, and $S = \pi R^2$ is the area of the inner cross-section of the artery. The change of flow is $\Delta q = \Delta(R^2\pi v) = \pi R^2 \Delta v$, which along with equation (1.19) gives

$$\Delta p = \left(\frac{\rho L}{\pi R^2}\right)\frac{\Delta q}{\Delta t}\,. \tag{1.20}$$

(h) One can conclude from part (e) that the change of pressure is analogous to the potential difference, while the flow of blood is analogous to the electrical current. Consequently equation (1.20) is analogous to the equation

$$\Delta \varphi = \mathcal{L}\frac{\Delta I}{\Delta t}\,, \tag{1.21}$$