Calculus

Calculus is an area of mathematics in which continuously changing values are studied, and it forms the basis of much of modern mathematics and its applications. It is introduced at the school level and students usually consider it as a set of tricks that they need to memorize. This book provides a thorough reintroduction to calculus, with an emphasis on logical development arising out of geometric intuition. It is written in a conversational style, with motivational discussions preceding every significant result or method. These features aid the student in acquiring the greater mathematical maturity required by a university course in calculus.

Calculus is intended for students pursuing undergraduate studies in mathematics or in disciplines like physics and economics where formal mathematics plays a significant role. For students majoring in mathematics, this book can serve as a bridge to real analysis. For others, it can serve as a base from where they can venture into various applications. After mastering the material in the book, the student would be equipped for higher courses both in the pure (real analysis, complex analysis) and the applied (differential equations, numerical analysis) directions.

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Amber Habib
Frontmatter
More Information

To the memory of my aunt and uncle

Shaista and Mohd Mohsin

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Contents

CAMBRIDGE

Cambridge University Press & Assessment
978-1-009-15969-2 — Calculus
Amber Habib
Frontmatter
More Information

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CAMBRIDGE

Cambridge University Press & Assessment
978-1-009-15969-2 — Calculus
Amber Habib
Frontmatter
More Information

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Preface

Even now such resort to geometric intuition in a first presentation of the differential calculus, I regard as exceedingly useful, from the didactic standpoint, and indeed indispensable, if one does not wish to lose too much time. But that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny.

— Richard Dedekind, 1872[1]

Calculus is a magical subject. A first encounter in school leads to a radical revision of one's ideas of what is mathematics. We are transported from a rather staid enterprise of counting and measuring to an adventure encompassing change, fluctuation, and a vastly increased ability to understand and predict the workings of the world. At the same time, the student encounters "magic" with both its connotations: awe and wonder on the one hand, mystery and a sense of trickery on the other. Calculus can appear to be a bag of tricks that are immensely useful, provided the apprentice wizard can perfectly remember the spells. As the student pursues mathematics further at university, her instructors may use courses in analysis to persuade her that calculus is a science rather than a mystical art. Alas, all too often the student perceives the new instruction as mere hair-splitting which gives no new powers and may even undermine her previous attainments. The first analysis course is for many an experience that makes them regret taking up higher mathematics.

This book is written to support students in this transition from the expectations of school to those of university. It is intended for students who are pursuing undergraduate studies in mathematics or in disciplines like physics and economics where formal mathematics plays a significant role. Its proper use is in a "calculus with proofs" course taught during the first year of university. The goal is to demonstrate to the student that attention to basic concepts and definitions is an investment that pays off in multiple ways. Old calculations can be done again with a fresh understanding that can not only be stimulating but also protects against error. More importantly, one begins to learn how knowledge can be extended to new domains by first questioning it in familiar terrain. For students majoring in mathematics, this book can serve as a

[1]Translated by W. W. Beman [6].

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978-1-009-15969-2 — Calculus
Amber Habib
Frontmatter
More Information

bridge to real analysis. For others, it can serve as a base from where they can make expeditions to various applications.

The origins of this book lie in our experiments with creating a bouquet of calculus courses at Shiv Nadar University. We found, in our initial years, that quite a few students with excellent marks in school leaving examinations were doing poorly in our introductory calculus course, even though it was not overly formal in its approach. The problem turned out to be that students considered themselves to already be experts and did not pay sufficient attention at the start of the course. Our solution was to rework the course to quickly take students out of their comfort zone and force them to explicitly rethink their assumptions. For example, putting integration at the start helped free them from the belief that it is a mere application of differentiation. Of course, there are other arguments for giving precedence to integration. The fact that it was understood two thousand years before differentiation has to stand for something. It suggests that the process of integration is easier to grasp intuitively, which is good reason to teach it first. As another example, the use of Tarski's version of the Completeness Axiom simplified many proofs and also gave a natural fit with the technique of nested intervals that we have used for the key theorems. In this way, over a few iterations, our shock tactics led us to a presentation with its own internal logic and advantages.

Much of the fascination of calculus comes from its various special topics. We have kept these out of the main presentation so that its structure is uncluttered. However, each chapter closes with one or two sets of thematic exercises, which develop either a particular application or further theory that takes an interested student closer to analysis. They have been given adequate structure and hints to ease the student's progress through them.

A textbook has to go beyond content, to attempt to influence the thinking habits of the reader. For this reason, small exercises called "Tasks" are scattered throughout the main text. These are intended to encourage the student to continually practice and develop her skills. After a while, she may recognize patterns in how each result leads to new questions, and be able to start posing her own questions. Each section is followed by a set of exercises, totalling about 400. Solutions, or hints towards solutions, are provided for all the odd numbered exercises at the end of the book. The solutions have been provided so that the problems do not become roadblocks, and also to provide examples of the expected level of reasoning and communication.

I would be be happy to respond to any corrections, suggestions, or requests for clarification. Please send an email to amberdevhabib@gmail.com.

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Amber Habib
Frontmatter
More Information

Acknowledgments

I must thank the two mathematicians who, thirty five years ago, reawakened in me a love for mathematics that had gone dormant under the pressure of crammed syllabi and exams: Shobha Madan and the late O. P. Juneja. I hope their methods of provoking students by well-timed questions have permeated this book, though I have no hope of capturing the accompanying smile or raised eyebrow. Later, I was fortunate to serve as a teaching assistant to Arthur Ogus, Alan Weinstein, Joseph Wolf, and Hung-Hsi Wu. While attending their lectures, and with much assistance from Vinay Kathotia, I realized that my love of analysis had not always improved my understanding of calculus. The bridges between the two realms have to be consciously built.

The driving force behind our experiments with teaching rigorous calculus at Shiv Nadar University was Debashish Bose, my co-instructor for most semesters since 2014. Sushil Singla went well beyond the expected contributions of a teaching assistant, and became a partner in recasting the course and giving form to this book. The supplementary exercises, on special topics, are to a large extent his doing.

The style of presentation is surely influenced by my experience of teaching in the Mathematical Training and Talent Search Programme (MTTS), and by the lectures and writings of S. Kumaresan, in particular. I have learned much about teaching from my fellow MTTS faculty, especially from S. D. Adhikari, A. J. Jayanthan, and S. Somasundaram. MTTS (https://mtts.org.in) is a remarkable programme that has lasted over a quarter of a century and has had significant success in changing students' views on mathematics and giving them the confidence to be mathematicians.

I am grateful to the team at Cambridge University Press for their encouragement, advice and support; especially to Qudsiya Ahmed, Vaishali Thapliyal, Vikash Tiwari and Aniruddha De. The anonymous referees also gave generously of their time and helped improve the exposition in several places.

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CAMBRIDGE
Cambridge University Press & Assessment
978-1-009-15969-2 — Calculus
Amber Habib
Frontmatter
More Information

Introduction

A calculus text written at this point of time can make its plea for existence on the novelty of its exposition and choice of content, rather than any originality in its mathematics. Let me state the case for this book. These explanations are primarily aimed at teachers, but students may also gain some perspective from them if they peruse them *after* going through at least the first two chapters.

The book aims to give a complete logical framework for calculus, with the proofs reaching the same levels of rigour as a text on real analysis. At the same time it eschews those aspects of analysis that are not essential to a presentation of calculus techniques. So it has the completeness axiom for real numbers, but not Cauchy sequences or the theorems of Heine–Borel and Bolzano–Weierstrass. At the other end of the spectrum, it omits the case for the importance of calculus through its applications to the natural sciences or to economics and finance.

The first chapter provides a description of real numbers and their properties, followed by functions and their graphs. For the most part, the material of this chapter would be known to students, but not in such an organized way. I typically use the first class of the course to ask students to share their thoughts on various issues. What are rational numbers? What are numbers? Which properties of numbers are theorems and which are axioms? What is the definition of a point or a line? What do we mean by a tangent line to a curve? These flow from one to another and from students' responses. By the end of the hour, with many students firm in their beliefs but finding them opposed just as firmly by others, I have an opportunity to propose that we must carefully put down our axioms and ways of reasoning so that future discussions may be fruitful. One may still ask whether the abstract approach is overdone; is it necessary to introduce general concepts like field and ordered field? The reason for doing so is that it provides a context within which simple questions can be posed and the student can practice creating and writing small proofs as a warm-up to harder tasks that wait ahead.

The first chapter is also where we introduce the completeness axiom, identified by Dedekind as the key property that sets real numbers apart and makes calculus a rich subject. We have used a version of the completeness axiom that was given by Tarski and is very close to Dedekind's original formulation, instead of the currently

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978-1-009-15969-2 — Calculus
Amber Habib
Frontmatter
More Information

popular least upper bound property. Tarski's version is easy to comprehend and simplifies many proofs.

In the second chapter we move directly to integration and its properties, preceding limits and continuity. This is in accord with both history and classic texts such as Apostol [2] and Courant and John [4]. The integration of monotone functions enables the early introduction and study of the logarithmic and exponential functions. The priority given to integration also enables the initial exposure to ϵ arguments to happen in a simplified setting. For example, to prove integrability or to compute an integral, for any $\epsilon > 0$ we have only to find a *single* pair of lower and upper sums that are within ϵ of each other. Whereas, in a limit argument, for each ϵ, one has to find δ such that good things happen for *every* point that is within δ of the base point. The use of Tarski's axiom also assists with the proofs of the properties of integration, giving them the form of direct manipulations of inequalities.

The third chapter takes up limits and continuity, with a fairly conventional presentation. We do make an attempt to confine ϵ-δ arguments to special cases where they are easier to handle, and then reduce the general case to the special case by other means. The major results (intermediate value theorem, boundedness theorem, extreme value theorem, small span theorem) are all proven along the same lines, via nested intervals. This should give the student a clearer sense of the unity of this material and make it easier to assimilate the proofs. With integration already in place, we are able to rigorously develop trigonometric functions. In this way, all the standard functions of calculus become available quite early.

The derivative finally enters the scene in the fourth chapter. We do not adopt the usual approach through tangent lines, as "tangent line" is not a well-defined notion at this stage. Instead, we introduce a "differentiable function" as one which is essentially linear locally. This leads quickly to the usual limit definition. Later in the chapter, we deviate again from the norm by proving results such as a zero derivative implying constancy via Fermat's theorem on local extremes, rather than as consequences of the mean value theorem. This quickens the development of curve sketching as well as techniques of integration.

The fourth chapter contains one half of the fundamental theorem of calculus: differentiation reverses integration. In the fifth, we obtain the other half: antiderivatives give values of definite integrals. This is the key to all techniques of integration. An instructor who is short on time can consider giving less attention to this chapter as students should be quite familiar with its contents and there are no surprises in its exposition. For our part, we have given a complete description of the technique of partial fractions, which is often only exhibited for low degree examples. The section on first-order differential equations raises issues of existence and uniqueness of solutions that students may not have considered earlier.

The place of the mean value theorem in a calculus course has been a matter of debate. The generic textbook gives it a central role. Yet many articles suggest that

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978-1-009-15969-2 — Calculus
Amber Habib
Frontmatter
More Information

the theorem is an obstacle for students and propose ways of bypassing it. We have taken a middle path and delayed its appearance until it is absolutely essential. When it does appear, in the sixth chapter, it is immediately put to serious work. It gives us L'Hôpital's rule, enables the use of Riemann sums to carry out surface area and volume calculations, and generates error estimates for Taylor approximations and for numerical integration. The delayed entrance is also a more dramatic one!

In the seventh chapter, we take up sequences and series. Here, we emphasize two techniques which are fundamental both for working out specific examples and proving general theorems. One is comparison with the geometric series. The other is viewing partial sums as lower/upper sums of corresponding integrals. Additionally, the chain of results starting with the Riemann rearrangement theorem and proceeding to the Cauchy product is obtained purely by the use of least upper bounds and simple algebra. These replace somewhat tricky ϵ arguments.

The eighth and final chapter first discusses power series and Taylor expansions. The functions we typically encounter in applications of calculus can be expanded as power series, and once this is done their differentiation and integration becomes easy. Various series can be summed by recognizing them as point values of power series. In the second half of this chapter, we take up two topics that set a path for future studies. Fourier series once caused a revolution in mathematics by forcing us to rethink our fundamental concepts such as sets and functions. They are now a unifying theme across many branches of mathematics and continue to inspire fresh developments. Finally, complex numbers explain many puzzles related to real numbers, and complex series lead to a deeper understanding of the relationship between differentiation and integration.

We have, now, described the core of the book. It can be covered in about 60 lecture hours. However, in many institutions only about 45 lecture hours may be available in a semester. In that case, the first six chapters would form a reasonable goal.

The sets of thematic exercises that follow each chapter can be used for self-study or for student presentations within a course. Instructors of related courses may also find them useful. For example, suppose an introductory course was being taught on ordinary differential equations (ODE). Typical texts for such courses concentrate on implementing techniques and skimp on proofs of existence and uniqueness of solutions. An instructor who wants to show that the standard solutions of a homogeneous second-order ODE with constant coefficients are *all* the solutions may struggle to find an approach that can be easily taught at this level. Such a proof is outlined in the exercises following Chapter 5. It needs only the result that a zero derivative implies constancy.

The exposition has benefited from the literature on alternative approaches to teaching calculus, especially the articles by Bers [48], Swann [58], Taylor [59] and Tucker [61]. A particular inspiration has been the text *Calculus Unlimited* by Marsden and Weinstein [23], which provoked the thought that a simplified approach could be

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978-1-009-15969-2 — Calculus
Amber Habib
Frontmatter
More Information

built around Tarski's formulation of the Completeness Axiom. Otto Toeplitz's *Calculus: A Genetic Approach* [35] convinced me that one should gradually increase generality, and at any stage use the generality that is appropriate. This led to the initial emphasis on monotonicity rather than continuity and enabled the rapid development of the standard transcendental functions. Other texts that have influenced this book are Apostol [2] and Spivak [30].

Signage

We have used the following icons to mark certain portions of the text:

- A digression or an issue that needs special attention.
- A motivational remark preceding a proof.
- □ End of an example.
- ■ End of a proof.

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