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Excerpt
More Information

1 | Real Numbers and Functions

Calculus can be described as the study of how one quantity is affected by another, focusing on relationships that are smooth rather than erratic. This chapter sets up the basic language for describing quantities and the relationships between them. Quantities are represented by numbers and you would have seen different kinds of numbers: natural numbers, whole numbers, integers, rational numbers, real numbers, perhaps complex numbers. Of all these, real numbers provide the right setting for the techniques of calculus and so we begin by listing their properties and understanding what distinguishes them from other number systems. The key element here is the completeness axiom, without which calculus would lose its power.

The mathematical object that describes relationships is called "function." We recall the definition of a function and then concentrate on functions that relate real numbers. Such functions are best visualized through their graphs, and this visualization is a key part of calculus. We make a small beginning with simple examples. A more thorough investigation of graphs can only be carried out after calculus has been developed to a certain level. Indeed, the more interesting functions, such as trigonometric functions, logarithms, and exponentials, require calculus for their very definition.

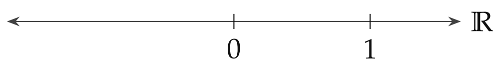
1.1 Field and Order Properties

We begin with a review of the set $\mathbb{R}$ of **real numbers**, which is also called the **Euclidean line**. It is a "review" in that we do not construct the set but just list its key attributes, and use them to derive others. For descriptions of how real numbers can be constructed from scratch, you can consult Hamilton and Landin [11], Mendelson [24], or most books on real analysis. The fundamental ideas underlying these constructions are easy to absorb, but the checking of details can be arduous. You would probably appreciate them more *after* reading this book.

What is the need for this review? Mainly, it is intended as a warm-up session before we begin calculus proper. Many intricate definitions and proofs lie in wait later, and we need to get ready for them by practising on easier material. If you are in a hurry and confident of your basic skills with numbers and proofs, you may skip ahead to the next section, although a patient reading of these few pages would also help in later encounters with linear algebra and abstract algebra.

Field Properties of Real Numbers

Any concept of "numbers" involves rules for combining them to create new ones. We shall use the term **binary operation** to denote a rule for associating a single member of a set to each pair of elements from that set.



The set $\mathbb{R}$ is equipped with two binary operations, $+$ (addition) and $\cdot$ (multiplication), and has two special elements named zero (0) and one (1), with the following fundamental properties:

R1. *Addition and multiplication are commutative:* $a+b=b+a$ *and* $a\cdot b=b\cdot a$ *for every* $a,b\in\mathbb{R}$.

R2. *Addition and multiplication are associative:* $a+(b+c)=(a+b)+c$ *and* $a\cdot(b\cdot c)=(a\cdot b)\cdot c$ *for every* $a,b,c\in\mathbb{R}$.

R3. *0 serves as identity for addition:* $0+a=a$ *for every* $a\in\mathbb{R}$.

R4. *1 serves as identity for multiplication:* $1\cdot a=a$ *for every* $a\in\mathbb{R}$.

R5. *Each* $a\in\mathbb{R}$ *has an additive inverse* $b\in\mathbb{R}$*, with the property* $a+b=0$.

R6. *Each non-zero* $a\in\mathbb{R}$ *has a multiplicative inverse* $c\in\mathbb{R}$*, with the property* $a\cdot c=1$.

R7. *Multiplication distributes over addition:* $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$ *for every* $a,b,c\in\mathbb{R}$.

The properties R1 to R7 are called the **field axioms** for $\mathbb{R}$. In general, if a set $\mathbb{F}$ has two binary operations $+$ and $\cdot$, such that these seven properties hold (with $\mathbb{R}$ replaced by $\mathbb{F}$ everywhere), then $\mathbb{F}$ is called a **field**. Other familiar examples of fields are the set $\mathbb{Q}$ of rational numbers and the set $\mathbb{C}$ of complex numbers. Each field has its own binary operations and its own special elements called zero and one.

Example 1.1.1

Consider the set $\mathbb{F}_2=\{0,1\}$ of just two elements named 0 and 1. Can we provide it with with binary operations $+$ and $\cdot$ such that it becomes a field in which 0 is the additive identity and 1 is the multiplicative identity? Well, since 0 is to be the additive identity, we must set

$$0+0=0,\quad 0+1=1+0=1.$$

What should be the additive inverse of 1? It obviously cannot be 0, so it must be 1. This gives $1+1=0$. Therefore $+$ is represented by following table.

$+$	0	1
0	0	1
1	1	0

Similarly, since 1 is to be the multiplicative identity, we must have

$$1 \cdot 1 = 1, \quad 1 \cdot 0 = 0 \cdot 1 = 0.$$

We also compute $0 \cdot 0$ as follows:

$$0 \cdot 0 = 0 \cdot 0 + 0 \cdot 1 = 0 \cdot (0 + 1) = 0 \cdot 1 = 0.$$

Hence $\cdot$ is represented by the following table.

$\cdot$	0	1
0	0	0
1	0	1

Let us verify that $\mathbb{F}_2$ really is a field. The commutativity of the two operations is clearly built into their definitions. We also see that 0 is the additive identity and 1 is the multiplicative identity. The additive and multiplicative inverses are also present. Only the associative and distributive laws have to be verified. Let $a, b, c \in \mathbb{F}_2$. Consider the following cases:

$$\begin{aligned}
a = 0 &\implies a + (b + c) = 0 + (b + c) = b + c = (0 + b) + c = (a + b) + c, \\
b = 0 &\implies a + (b + c) = a + (0 + c) = a + c = (a + 0) + c = (a + b) + c, \\
c = 0 &\implies a + (b + c) = a + (b + 0) = a + b = (a + b) + 0 = (a + b) + c, \\
a = b = c = 1 &\implies a + (b + c) = 1 + (1 + 1) = (1 + 1) + 1 = (a + b) + c.
\end{aligned}$$

So + is associative. We ask you to verify in a similar fashion the associativity of $\cdot$ as well as the distributive law. □

The set of non-zero real numbers is denoted by $\mathbb{R}^*$. We shall usually abbreviate $a \cdot b$ to ab.

Theorem 1.1.2

The field $\mathbb{R}$ has the following properties.

1. *0 is the only additive identity and 1 is the only multiplicative identity.*
2. *The additive inverse of any real number is unique.*
3. *The multiplicative inverse of any non-zero real number is unique.*

The important thing is to realize that these claims need proof, and then to prove them using only the field axioms.

Proof. Suppose $0'$ and $1'$ are additive and multiplicative identities, respectively. Then we have

$$\begin{aligned}
0' &= 0 + 0' && \text{(because 0 is an additive identity)} \\
&= 0 && \text{(because } 0' \text{ is an additive identity).}
\end{aligned}$$

Similarly, we have $1' = 1' \cdot 1 = 1$. This shows the uniqueness of 0 and 1 as identities for addition and multiplication.

Next, suppose a has additive inverses b and c. Then $a+b=0$ and $a+c=0$. Hence,

$$b = b + 0 = b + (a+c) = (b+a) + c = 0 + c = c.$$

This shows the uniqueness of the additive inverse. You can similarly show the uniqueness of the multiplicative inverse. ∎

Having established that the inverses are unique, we can give them special names. We shall denote the additive inverse of a by $-a$ and the multiplicative inverse by $1/a$ or a^{-1}.

Theorem 1.1.3 (Cancellation Laws)

Let $a, b, c \in \mathbb{R}$. Then the following hold:

1. *If $a+b=a+c$ then $b=c$.*
2. *If $ab=ac$ and $a \neq 0$ then $b=c$.*

Proof. The cancellation laws are based on associativity and the existence of inverses.

$$\begin{aligned} a+b=a+c &\implies (-a)+(a+b) = (-a)+(a+c) && \text{(existence of inverse)} \\ &\implies ((-a)+a)+b = ((-a)+a)+c && \text{(associativity)} \\ &\implies 0+b = 0+c && \text{(property of inverse)} \\ &\implies b = c && \text{(property of identity).} \end{aligned}$$

If $a \neq 0$ then it has a multiplicative inverse a^{-1} and we have

$$\begin{aligned} ab = ac &\implies a^{-1}(ab) = a^{-1}bc \implies (a^{-1}a)b = (a^{-1}a)c \\ &\implies 1 \cdot b = 1 \cdot c \implies b = c. \end{aligned}$$

You should provide the justification for each step, as we had done for the case of addition. ∎

Theorem 1.1.4

Let $a, b, c \in \mathbb{R}$. Then the following hold:

1. $0 \cdot a = 0$.
2. $-(-a) = a$.
3. *If $a \in \mathbb{R}^*$ then $(a^{-1})^{-1} = a$.*
4. $(-1)a = -a$.
5. $(-1)(-1) = 1$.

6. $(-a)(-b) = ab$.
7. *If* $ab = 0$ *then* $a = 0$ *or* $b = 0$.

Proof.

1. Use $0 = 0 + 0$:
$$a \cdot 0 = a \cdot (0+0) = (a \cdot 0) + (a \cdot 0) \implies 0 + (a \cdot 0) = (a \cdot 0) + (a \cdot 0) \implies 0 = a \cdot 0.$$
2. If we let $b = -(-a)$ we have $b + (-a) = 0$. We also have $a + (-a) = 0$. We apply cancellation to get $b = a$.
3. Let $b = (a^{-1})^{-1}$, so that $a^{-1}b = 1$. We also have $a^{-1}a = 1$. Apply cancellation.
4. We have to show that $(-1)a$ is the additive inverse of a. So we add them:
$$(-1)a + a = (-1) \cdot a + 1 \cdot a = ((-1) + 1) \cdot a = 0 \cdot a = 0.$$
5. Substitute $a = -1$ in the previous statement.
6. $(-a)(-b) = \Big(a(-1)\Big)\Big((-1)b\Big) = a\Big((-1)\Big((-1)b\Big)\Big) = a\Big(\Big((-1)(-1)\Big)b\Big) = a(1 \cdot b) = ab$.
7. We will show that if $a \neq 0$ then we must have $b = 0$. So at least one of $a = 0$ and $b = 0$ must hold.
$$a \neq 0 \implies a^{-1}(ab) = a^{-1}0 \implies (a^{-1}a)b = 0 \implies 1 \cdot b = 0 \implies b = 0.$$

■

Task 1.1.5

Verify that $-(a+b) = (-a) + (-b)$ *and* $(ab)^{-1} = a^{-1}b^{-1}$.

For any $a, b \in \mathbb{R}$, the sum $a + (-b)$ is denoted by $a - b$ and is called the **difference** of a and b. The process of obtaining $a - b$ is called **subtraction**. Similarly, if $b \in \mathbb{R}^*$, the product $a \cdot (1/b)$ is denoted by $\frac{a}{b}$ or a/b and is called the **ratio** of a and b. The process of obtaining a/b is called **division**.

The **square** of a number x is its product with itself and is denoted by x^2.

Task 1.1.6

Show that $(-x)^2 = x^2$.

Task 1.1.7

Use the field axioms of $\mathbb{R}$ *to prove the following:*

(a) $-\dfrac{a}{b} = \dfrac{-a}{b} = \dfrac{a}{-b}$ *if* $b \neq 0$,

(b) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$ *if* $b, d \neq 0$.

Order Properties of Real Numbers

Since the field axioms of $\mathbb{R}$ are also satisfied by $\mathbb{Q}$ and $\mathbb{C}$, we know that they do not completely determine the real numbers. What else is special about $\mathbb{R}$?

The non-zero real numbers $\mathbb{R}^*$ split into two types: **positive** and **negative**. We shall denote the set of positive real numbers by $\mathbb{R}^+$ and the set of negative real numbers by $\mathbb{R}^-$. The key facts associated with this split are as follows.

R8. *Every non-zero real number is either positive or negative.*

R9. *Zero is neither positive nor negative.*

R10. *No real number is both negative and positive.*

R11. *A real number is negative if and only if its additive inverse is positive.*

R12. *The sum and product of positive numbers are positive.*

Complex numbers cannot be split into positive and negative ones in this manner. For, one of $\pm i$ would be positive, as well as one of ± 1. Hence both $-1 = i^2 = (-i)^2$ and $1 = 1^2 = (-1)^2$ would be positive!

The properties R8 to R12 are called the **order axioms** of $\mathbb{R}$. Let us see some of their consequences.

Theorem 1.1.8

1. *If $x, y \in \mathbb{R}^-$ then $x + y \in \mathbb{R}^-$.*
2. *If $x, y \in \mathbb{R}^-$ then $xy \in \mathbb{R}^+$.*
3. *If $x \in \mathbb{R}^+$ and $y \in \mathbb{R}^-$ then $xy \in \mathbb{R}^-$.*
4. *If $x \in \mathbb{R}^*$ then $x^2 \in \mathbb{R}^+$.*
5. $1 \in \mathbb{R}^+$.

Again, these are familiar properties, which you were asked to memorize in school. We wish to convert them to proven facts. We treat the first two to show you the way, and leave the others as exercises.

Proof.
$$\begin{aligned} x,y \in \mathbb{R}^- &\implies -x,-y \in \mathbb{R}^+ \implies (-x)+(-y) \in \mathbb{R}^+ \\ &\implies x+y = -((-x)+(-y)) \in \mathbb{R}^-. \\ x,y \in \mathbb{R}^- &\implies -x,-y \in \mathbb{R}^+ \implies (-x)(-y) \in \mathbb{R}^+ \\ &\implies xy = (-x)(-y) \in \mathbb{R}^+. \end{aligned}$$
∎

The split into positive and negative allows us to think of larger and smaller real numbers (an “ordering”) as follows. We say that a is **greater** than b, denoted by $a > b$, if $a - b \in \mathbb{R}^+$. In this case, we also say that b is **less** than a and denote that by $b < a$.

Theorem 1.1.9

Let $a, b, c \in \mathbb{R}$. Then the following hold.

1. $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ *and* $\mathbb{R}^- = \{x \in \mathbb{R} \mid x < 0\}$.
2. *(Trichotomy) Exactly one of the following holds:* $a = b$ *or* $a > b$ *or* $a < b$.
3. *(Transitivity) If* $a > b$ *and* $b > c$ *then* $a > c$.
4. *If* $a > b$ *then* $a + c > b + c$.
5. *Let* $c > 0$. *If* $a > b$ *then* $ac > bc$.
6. *Let* $c < 0$. *If* $a > b$ *then* $ac < bc$.
7. *If* $a < b$ *then* $a < \dfrac{a+b}{2} < b$.
8. *If* $0 < a < b$ *then* $0 < 1/b < 1/a$.
9. *Suppose* $a, b > 0$. *Then* $a > b \iff a^2 > b^2$.
10. *Suppose* $a, b > 0$. *Then* $a = b \iff a^2 = b^2$.

Proof.

1. We have $x > 0 \iff x - 0 \in \mathbb{R}^+ \iff x \in \mathbb{R}^+$. We similarly obtain the description of $\mathbb{R}^-$.
2. First, we note that $a = b$ implies $a - b = 0$, which rules out $a > b$ as well as $a < b$.

   Now let a, b be distinct real numbers. We have to prove that exactly one of $a > b$ and $a < b$ holds. Since a, b are distinct, $a - b \neq 0$. Therefore $a - b$ belongs to exactly one of $\mathbb{R}^+$ and $\mathbb{R}^-$. Now $a - b \in \mathbb{R}^+$ corresponds to $a > b$ and $a - b \in \mathbb{R}^-$ corresponds to $a < b$.
3. Hint: Consider $a - c = (a - b) + (b - c)$.
4. Hint: Consider $(a + c) - (b + c) = a - b$.
5. Hint: Consider $ac - bc = (a - b)c$.
6. As above.
7. Hint: Add a to both sides of $a < b$ to get one of the inequalities. Add b instead to get the other.
8. Hint: $\dfrac{1}{a} - \dfrac{1}{b} = \dfrac{b-a}{ab}$.
9. Hint: $b^2 - a^2 = (b - a)(b + a)$.
10. As above. ∎

Note that item 7 of this theorem implies that there are infinitely many real numbers between any two distinct ones.

Let A be a subset of $\mathbb{R}$.

- An element $M \in A$ is called the **maximum** of A if $a \leq M$ for every $a \in A$. We write $M = \max(A)$.
- An element $m \in A$ is called the **minimum** of A if $m \leq a$ for every $a \in A$. We write $m = \min(A)$.

A maximum element is also called **greatest** while a minimum element is also called **least**.

Example 1.1.10

Let $A = \{x \in \mathbb{R} \mid x \leq 1\}$. Then 1 is the maximum of A. ☐

Task 1.1.11

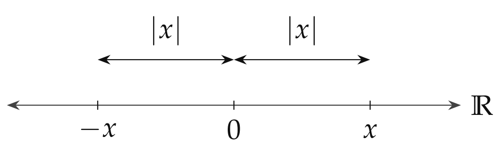
Let $A = \{x \in \mathbb{R} \mid x < 1\}$. Show that A has no maximum.

Absolute Value

Let us continue this overview of familiar facts about the real numbers by recalling the definition of the **absolute value** of a real number x,

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

We think of a real number as having two aspects: a *direction* determined by whether it is positive or negative, and a *magnitude* given by its absolute value.



Theorem 1.1.12

Let $x, y \in \mathbb{R}$. Then we have the following.

1. $|x| \geq 0$.
2. $|x| = 0$ *if and only if* $x = 0$.
3. $|x^2| = |x|^2 = x^2$.
4. $|xy| = |x||y|$.
5. *(Triangle Inequality)* $|x + y| \leq |x| + |y|$.
6. $|x - y| \geq ||x| - |y||$.

Proof. The first two claims are obvious from the definition. Proofs for the others are given below. We make use of the fact that if $a,b \geq 0$ then $a = b \iff a^2 = b^2$.

3. Since $x^2 \geq 0$, we have $|x^2| = x^2$. Further,

$$|x|^2 = \begin{cases} x^2 & \text{if } x \geq 0 \\ (-x)^2 & \text{if } x < 0 \end{cases} = x^2.$$

4. $|xy|^2 = (xy)^2 = x^2y^2 = |x|^2|y|^2 = (|x||y|)^2$.

5. $|x+y|^2 = (x+y)^2 = x^2 + y^2 + 2xy \leq |x|^2 + |y|^2 + 2|x||y| = (|x| + |y|)^2$.

6. $|x-y|^2 = (x-y)^2 = x^2 + y^2 - 2xy \geq |x|^2 + |y|^2 - 2|x||y| = (|x| - |y|)^2 = ||x| - |y||^2$. ■

Task 1.1.13
For any $x,a \in \mathbb{R}$ with $a \geq 0$, prove that $|x| \leq a \iff -a \leq x \leq a$.

Since we think of $|x|$ as the magnitude or size of a real number, $|x-y|$ becomes a measure of the gap between x and y. We call it the **distance** between x and y. The properties of absolute value convert to the following properties of distance.

Theorem 1.1.14
Let $x,y,z \in \mathbb{R}$. Then we have the following:

1. *(Positivity) $|x-y| \geq 0$, and $|x-y| = 0$ if and only if $x = y$.*
2. *(Symmetry) $|x-y| = |y-x|$.*
3. *(Triangle Inequality) $|x-z| \leq |x-y| + |y-z|$.*

Proof. This is left as an exercise for you. ■

Types of Real Numbers

The set of real numbers includes various special types of numbers:

- By repeatedly adding 1 we generate the subset of **natural numbers**,

  $$\mathbb{N} = \{1, 2 = 1+1, 3 = 2+1, \ldots\}.$$

  By combining (5) of Theorem 1.1.8 and (4) of Theorem 1.1.9 we see that $1 < 2 < 3 < \cdots$.

- By including zero with natural numbers we get **whole numbers**,

  $$\mathbb{W} = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}.$$

- By further including the additive inverse of each whole number we get **integers**,

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

- By dividing integers with each other we get **rational numbers**,

$$\mathbb{Q} = \{\, a/b \mid a, b \in \mathbb{Z} \text{ and } b \neq 0 \,\}.$$

Some examples of rational numbers are $\frac{8}{6}$, $\frac{12}{9}$, $\frac{-4}{3}$, $\frac{4}{-3}$, and $\frac{2}{1}$.

Task 1.1.15

Let $a, b, c, d \in \mathbb{Z}$ with $b, d \neq 0$. Show that $\frac{a}{b} = \frac{c}{d} \iff ad = bc$.

The positive rational numbers will be denoted by $\mathbb{Q}^+$. Any $x \in \mathbb{Q}^+$ can be expressed as $x = p/q$ with $p, q \in \mathbb{N}$.

Some mathematicians include 0 in the set of natural numbers itself. So be careful when you see someone using $\mathbb{N}$, and check whether or not they include 0.

Those real numbers that are not rational numbers are called **irrational numbers**. At this point in this text, we still do not know enough about real numbers to be able to say whether there are any irrational numbers! This will be clarified in the next section.

Mathematical Induction

We make a small digression to recall some important facts about natural numbers.

Principle of Mathematical Induction: If A is a subset of $\mathbb{N}$ that contains 1 and is closed under adding 1 then $A = \mathbb{N}$. Alternately: If $P(n)$ is a statement about n (for every natural number n) such that $P(1)$ is true and the truth of $P(n)$ implies the truth of $P(n+1)$, then $P(n)$ is true for every natural number n.

Mathematical induction is used to prove statements that hold for every natural number. As an example, we will use it to better understand integer powers of real numbers.

Example 1.1.16

We define **integer powers** as follows: First, we define $x^0 = 1$ for any $x \in \mathbb{R}$. Then, for any $n \in \mathbb{N}$, we define $x^n = x \cdot x^{n-1}$. If $x \neq 0$, we define $x^{-n} = (x^n)^{-1}$. We will use induction to prove the following:

$$\text{If } x \neq 0 \text{ and } n \in \mathbb{N} \text{ then } (x^{-1})^n = (x^n)^{-1}.$$

Let $P(n)$ be the statement that $(x^{-1})^n = (x^n)^{-1}$. Then $P(1)$ is the statement $x^{-1} = x^{-1}$, which is certainly true. Now assume that some $P(n)$ is true (this is called the