Actions of Groups

Using the unifying notion of group actions, this second course in modern algebra introduces the deeper algebraic tools needed to get into topics only hinted at in a first course, like the successful classification of finite simple groups and how groups play a role in the solutions of polynomial equations. Because groups may act as permutations of a set, as linear transformations on a vector space, or as automorphisms of a field, the deeper structure of a group may emerge from these viewpoints, two different groups can be distinguished, or a polynomial equation can be shown to be solvable by radicals. By developing the properties of these group actions, readers encounter essential algebra topics like the Sylow Theorems and their applications, Galois theory, and representation theory. Warmup chapters that review and build on the first course and active learning modules help students transition to a deeper understanding of the ideas.

JOHN McCLEARY is Professor of Mathematics on the Elizabeth Stillman Williams Chair at Vassar College. He received his PhD in mathematics from Temple University, where he completed a thesis on algebraic topology under the direction of James Stasheff. The author of several books, including *A User's Guide to Spectral Sequences and Geometry from a Differentiable Viewpoint*, McCleary has published extensively on topology, the history of mathematics, and other topics. His current research focuses on algebraic topology – specifically, where algebra reveals what topology may conceal.

This book contains the perfect mix of all the ideas central to a second course in algebra. The focus on group actions will serve any student well as they continue their study of mathematics. The text goes at a good pace and the "Get to know" exercises sprinkled throughout the chapters provide excellent opportunities for students to pause, work with ideas presented, and master the content. This textbook is definitely unique, and I have not seen its equal.

JOSEPH KIRTLAND, Marist College

When I was young, I really enjoyed my first class in group theory, spending delightful (and at times frustrating) hours at a blackboard in an empty classroom trying to solve the homework problems. But I really didn't see why groups were all that important in the rest of mathematics. Yes, I got that the two main examples of groups (permutation groups and invertible matrices) were natural to study, and I heard my professor tell us that groups permeate modern mathematics. But I didn't see it (though I was willing to believe that my professor was probably right.) This is one of the books I wish I had had, as it lets people start down the road to understanding that groups are indeed one of the unifying concepts of modern mathematics.

THOMAS GARRITY, Williams College

This book provides a wealth of topics building on what is likely to be available to students after a first course on basic abstract algebra. A major theme is that of "groups in action," and most of the topics involve groups in a leading role (e.g., Galois theory and representation theory). The book would certainly be a useful text for a course, and also provide independent reading for a motivated student. It should also provide a good preparation or revision before deeper study at graduate student level.

ANDREW J. BAKER, University of Glasgow

Actions of Groups

A Second Course in Algebra

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To my students

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Preface

> For, all that creates the beauty and at the same time the difficulty of this theory is that one has ceaselessly to indicate the path of the analyses and to anticipate the results, without ever being able to carry them out.
>
> ÉVARISTE GALOIS, *Discours préliminaire,* IX.1830

For students of mathematics, the first course in Modern Algebra is an invitation to a new world. Visiting any place for the first time, one wants to see the grand sights, taste the local cuisine, and master a bit of the local language. All in a semester. Of course, that leaves a lot of places to come back to and deeper dives to make. This book is an answer to the question "What should I visit next?"

The standard first course in Algebra introduces groups and the fundamental concepts of subgroups, homomorphisms and isomorphisms, normal subgroups and quotient groups, and product groups, up to the Fundamental Theorem of Finite Abelian Groups. Rings are next with the analogous fundamental concepts of subrings and ideals, ring homomorphisms and isomorphisms, quotient rings, leading up to the notion of divisibility, especially in polynomial rings. If time remains (it often does not), then fields and their extensions are introduced, together with finite fields. The promise of a synthesis of these ideas in Galois theory is often made. Many second courses in Algebra focus on this beautiful topic.

Galois theory is difficult to teach (see the epigram of Galois that precedes this Introduction) and to learn. The accumulation of insights against the less sophisticated background of the first course grows into a labyrinth for students in which there are unmarked forks in the path without clear motivations for choice.

When I was a graduate student, the buzz at many colloquium talks was the number

$$808\,017\,424\,794\,512\,875\,886\,459\,904\,961\,710\,757\,005\,754\,368\,000\,000\,000$$
$$= 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71,$$

which is the order of the *Monster*, a simple group whose existence would complete the project of the classification of finite simple groups. It was conjectured independently in 1973 to exist by BERND FISCHER and ROBERT GRIESS, and many things were known about it before its existence, such as its 194×194 character table [15]. How could such numbers like the order be found? Why is the monster the largest and last sporadic simple group? What was this character table, and what did it have to do with the existence of the group? The announcement in 1981 of a construction of the Monster by Griess [36] excitedly brought the classification of finite simple groups to completion. How can that achievement be understood?

After a conversation with a colleague (from Williams College), I decided to veer off the usual path and teach the second course in Algebra at Vassar College on representation theory. The beauty of these ideas rivaled that of Galois theory, and I began to get s small idea of how representations could be used in the study of finite groups. The representation theory provides a bridge from the use of group actions on sets via representations on vector spaces to groups acting on field extensions. This point of view is the basis for this book.

The story comes in three parts: Groups act on sets, groups act on vector spaces, and groups act on field extensions. In the first part, I provide a warmup chapter that focuses on the Jordan–Hölder Theorem, which associates to a finite group a list of simple groups that can be assembled into a given group. The notion of a solvable group is introduced, and the family of alternating groups A_n for $n \geq 5$ are shown to be simple groups. Some of the this material may have been taught in the first course and may be reviewed as a warmup. The second chapter treats groups acting on sets, from the familiar appearance of the symmetries of a geometric figure (the dihedral groups) to groups acting on various sets constructed from the group itself. This rich subject includes the notion of a transitive subgroup of a permutation group that plays an important role in Galois theory. The Sylow Theorems are the focus of Chapter 2, where they are applied to determine if a group of a particular cardinality can be a simple group. These theorems are also a nice warmup to Galois theory, which makes the connection between permutations of sets and the eventual use of permutations of roots of a polynomial. I added two important theorems to this discussion, *Iwasawa's Lemma*, which gives a criterion for simplicity, and the

Burnside Transfer Theorem, which provides a sophisticated tool for studying whether a group of a given order is simple or not.

Chapter 3 is a warmup chapter on linear algebra. Most students who have had the first course in Modern Algebra have been introduced to linear algebra in an earlier course, and some may have had a second course in Linear Algebra as well. In this chapter the useful notion of a projection is introduced along with tensor products of vector spaces. I introduce Hermitian inner products on complex vector spaces in anticipation of their appearance in representation theory. I also consider matrix groups over fields to obtain another class of simple groups, $\mathrm{PSl}_n(\mathbb{F})$ for $n \geq 2$, which give finite simple groups when $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$ for p a prime. Chapter 4 focuses on representation theory, developed in the framework of finite groups acting on vector spaces. There is an alternative viewpoint that features modules over the group ring $\mathbb{C}G$. Sadly, this approach is left out in an effort to work with as few prerequisites as possible. Comments on this choice are found in the Epilogue. The beautiful cascade of constructions and results is pointed toward obtaining properties of a finite group that reveal deeper structure and the connections between subgroup data, quotient data, and the invariants of a given group. The example of A_5 is developed completely.

Chapter 5 treats fields and polynomials. This chapter may have the largest overlap with the first course. The basic notion of a field extension is introduced, including the key notion of a splitting field. The existence of algebraic closures is proved, and further properties of finite fields are proved including the Frobenius automorphism. With the existence of new finite fields, there come new finite simple groups, $\mathrm{PSl}_n(\mathbb{F}_q)$. The exercises in this chapter treat the classical problem of compass and straightedge constructions in geometry. Chapter 6 focuses on Galois theory, which is about the isomorphisms between fields. The emphasis up to this chapter on the actions of groups gives motivation for the path of development of Galois's (and Dedekind's) important ideas. I cover the classical topics of the Galois correspondence, cyclotomic extensions, solvability by radicals, including the cases of cubic and quartic polynomials. The case of quintic polynomials is discussed from the point of view of the inverse Galois problem – which transitive subgroups of the symmetric group on five letters are realized as a Galois group of an extension over the rationals? Galois theory also provides some control of algebraic integers, and I return to a celebrated result of Burnside, which combines representation theory and Galois theory. The Normal Basis Theorem provides another bridge between representation theory and Galois theory establishing the insight of Emmy Noether and Max Deuring that Galois extensions realize the regular representation of the Galois group.

Some sections of the book involve fairly sophisticated topics that can be skipped on first reading. These sections are marked with an asterisk.

In all my courses of the past few years I have introduced a regular active learning session, my *Workshop Wednesdays*, in which students work together on some problems. My goal is to offer additional opportunities to explore the topics of the week, as well as the opportunity to talk together about mathematics. If you have studied a foreign language, then these sorts of sessions build assurance in knowledge of vocabulary, syntax, and semantics. All of this is useful in learning new mathematics. I have built this idea into the book as sets of exercises usually labeled *Get to know* something. These sets of exercises come at points in the narrative where hands-on experience is important. I encourage the reader to keep a pad and writing instrument handy to try their hand in the moment.

How to use this book

First courses in Algebra vary widely. Depending upon the coverage of topics (Did the course focus only on groups? Were the Sylow Theorems part of the curriculum? How much ring theory and field theory were covered? Were vector spaces over an arbitrary field part of the linear algebra course?), the reader will find plenty that they might skip or review, especially in the warmup chapters. Alternatively, the warmup chapters may be assigned as supplemental reading, or as in-class short presentations. The heart of the matter lies in the even numbered chapters. A thirteen-week course might be arranged in the following (breakneck speed) breakdown by week:

1. The Jordan–Hölder Theorem, simple and solvable groups, and A_n is simple for $n \geq 5$.
2. Action of a group on a set, orbits, stabilizers, equivariant mappings, the Class Equation.
3. Transitive actions, p-groups, the Sylow Theorems and consequences.
4. The Fundamental Theorem of Linear Algebra via projections, tensor products.
5. Hermitian vector spaces, matrix groups, $\mathrm{PSl}_n(\mathbb{F})$ is simple for most n.
6. Representations, subrepresentations, irreducible representations, Maschke's Theorem.
7. Schur's Lemma, finding irreducible representations, characters.
8. The orthogonality theorems for character tables. Lifts, kernels, linear representations.

9. Field extensions, field homomorphisms, splitting fields, Kronecker's Theorem.
10. Review irreducibility criteria, finite fields and the Frobenius automorphisms, existence of algebraic closures.
11. Field automorphisms, finite field extension example, Galois group, Galois correspondence.
12. Dedekind Independence Theorem, normal and Galois extensions, the Fundamental Theorem of Galois Theory.
13. Cyclotomic extensions, solvability by radicals and Galois' Theorem.

Making some of the topics into workshops can preserve the order while encouraging active learning. The proofs of many results may be skipped to present the architecture of the ideas more clearly, but certainly many should be presented to reveal the inner workings of the subjects. I am sure every reader and every instructor will make their own choices.

Acknowledgments

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