'The classical mechanics of systems of finitely many point particles belongs to the bedrock of theoretical physics every physicist has to be familiar with. Giorgilli's book on Hamiltonian mechanics is a treasure chest. It conveys the author's profound knowledge of the history of the subject and provides a pedagogical exposition of the basic mathematical techniques. Very many important examples of systems are discussed, and the reader is guided towards fairly recent and new developments in this important subject. This book will become a classic.'

— Jürg Fröhlich, *ETH Zürich*

'This is an impressive book by one of the protagonists of the modern theory of dynamical systems. It contains the basic steps of the Hamiltonian theory, but it emphasizes the modern developments of the theory that started with Poincaré and Birkhoff, reaching the study of chaos. The book has two important advantages. It gives the successive steps needed by a beginner who enters into this field, up to its most recent developments. At the same time it provides a remarkable historical account of the developments of the theory. For example, the author uses the Lie series instead of the previous cumbersome canonical transformations, he gives a new classical proof of the Kolmogorov–Arnold–Moser theorem, he considers exponential and superexponential stability and so on. Giorgilli introduced many new ideas in the theory of dynamical systems, but he presents the new developments in a systematic way without emphasizing his own contributions. Finally, the book contains an extensive list of useful references. This book will be of great value for anyone interested in dynamics, and it is absolutely necessary for any library in physics, astronomy and related fields.'

— George Contopoulos, *Academy of Athens*

'This is a book that the reader will refer to over and over again: it provides a theoretical and practical framework for understanding Hamiltonian formalism and classical perturbation theory. It contains a readable complete proof of the most important results in the field (Kolmogorov–Arnold–Moser theorem and Nekhoroshev's theorem) as well as their applications to the fundamental problems of celestial mechanics. It also explains how small divisors are at the origin of the divergence of perturbation series and Poincaré's discovery of homoclinic intersections and of chaotic behaviour in near-to-integrable systems. What a remarkably useful and exciting book!'

— Stefano Marmi, *Scuola Normale Superiore, Pisa*

'This amazing book, written by a prominent master of the theory of Hamiltonian systems, is a wonderful gift for anyone interested in classical dynamics, from a novice student to a sophisticated expert. The author, together with the reader, goes from the definition of canonical equations to such shining peaks as the Kolmogorov theorem on invariant tori and the Nekhoroshev theorem on exponential stability (with complete proofs, and for each of these fundamental theorems two entirely different proofs are presented!). Carefully selected examples and exercises and historical digressions greatly facilitate learning the material and turn reading the book into an intellectual festivity.'

— Mikhail Sevryuk, *Semenov Federal Research Center of Chemical Physics, Moscow*

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Notes on Hamiltonian Dynamical Systems

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Apology

Durissima est hodie conditio scribendi libros Mathematicos, præcipue Astronomicos. Nisi enim servaveris genuinam subtilitatem propositionum, instructionum, demonstrationum, conclusionum, liber non erit Mathematicus: sin autem servaveris, lectio efficitur morosissima, præsertim in Latina lingua, quæ caret articulis, & illa gratia quam habet Græca, cum per signa literaria loquitur.

Johannes Kepler, *Astronomia Nova*

This book is a somewhat reordered collection of notes accumulated over the years while I was preparing my lectures at different levels, ranging from second-year faculty courses on Mechanics, to more advanced courses on Celestial Mechanics and Dynamical Systems, to series of lectures at the postgraduate and PhD level on perturbation methods. I should stress that it is neither a treatise nor an encyclopedic collection: the objective is mainly didactical. The guiding line is to make a journey from a basic knowledge of the Hamiltonian formalism, without assuming it to be known in advance to the reader, up to some most recent results in the field of Dynamical Systems, in particular in the Hamiltonian framework.

The choice of the arguments reflects my personal experience, both in research and in teaching; therefore it is unavoidably incomplete. The best way to teach, in my modest opinion, is to respect the historical development, whenever possible – which I did only partially. I tried to follow the development of our knowledge starting with the dream of the nineteenth century – to find an effective way to solve the problems of Mechanics and in particular of the dynamics of the Planetary System – and ending with the amazing coexistence of order and chaos that has been discovered by Poincaré, but has progressively become known only in the second half of the past century and nowadays is an inexhaustible argument of research and of many books and conferences. The style of exposition is that of a lecture or a talk; I attempt to place the mathematical results in the context (in a broad sense, not merely mathematical) of the problems that they answer.

All this may explain the perhaps surprising epigraph at the beginning of this apology. The idea, to quote the sentence of Kepler – with all due respect to that great mathematician and astronomer, if he cares to accept my apologies – is closely linked to the style of the presentation. The text mixes technical sections, including formal statements and detailed proofs of the main theorems, with more discursive parts including the results of numerical experiments and, here and there, some historical digression; in this sense it is not written as a typical mathematical book. So to speak, I tried to use theorems in order to explain the outcome of numerical experiments, and numerical experiments in order to suggest how to interpret the theorems. At the same time, I used history in order to instill a feeling that our current knowledge is only a provisional step on a long way paved with problems and questions, successes and failures, moments of enthusiasm or frustration, small steps forward and, from time to time, some profound intuition or brilliant discovery that we owe to great scientists and which indicates the way for the future.

A last comment on the epigraph. A reader who considers the reference to Latin and Greek languages as outdated is definitely right; but let me suggest to replace *Latina lingua* with *'CTL'* (Common Technical Language, that funny communication tool, resembling English, which is popular in the scientific community), and to replace *Græca* with the reader's own native or preferred language.

My hope is that the reader – a student, say – will find this book a companion and a useful guide in the journey that I propose. But it is not my intention to invite the reader – in particular a reader wanting to spend part of his or her life in scientific research – to stop at the end of the book. I will consider my objectives as fully reached when the reader concludes that my notes are too limited and casts this book into some trunk in the attic, looking forward to something better or perhaps to adding a next step.

Locutus sum in corde meo dicens:
ecce magnus effectus sum
et præcessi sapientia omnes
qui fuerunt ante me in Hierusalem,
et mens mea contemplata est multa sapienter et didicit,
dedique cor meum ut scirem prudentiam
atque doctrinam erroresque et stultitiam,
et agnovi quod in his quoque esset labor et adflictio spiritus,
eo quod in multa sapientia multa sit indignatio,
et qui addit scientiam addit et laborem.

Qohelet 1, 16–18

Plan of the Book

The present notes may be considered as structured in three parts with an *intermezzo* – like an *opera*.

The first part is composed of Chapters 1 and 2. The arguments covered are essentially the ones developed in the second half of the nineteenth century, up to Poincaré.

Chapter 1 is like an *overture*; the Hamiltonian formalism is introduced from scratch. It is addressed mainly to undergraduate faculty students, assuming that they have a basic knowledge of Analysis and of the Lagrangian methods of Mechanics, but not necessarily of the Hamiltonian formalism. Readers who are already familiar with the canonical formalism are encouraged to skip this chapter.

Chapter 2 deals with canonical transformations. The aim is to provide the practical tools for the rest of the book; therefore it intends to go a little beyond a short exposition such as can be usually found in many first-level textbooks on Mechanics; at the same time there is no pretension to completeness. The purpose is to present the argument in a self-consistent form and in traditional terms, so that the exposition is reasonably simple – at least according to the author's experience.

The second part consists of three chapters. It is concerned with the transition from the dream of solving the equations of any mechanical system to the discovery that this was only an unachievable dream.

Chapter 3 is devoted to classical integration methods based on use of first integrals. Its historical location begins with the middle of the nineteenth century, with the theorem of Liouville, and extends till a century later, jumping to the theorem of Arnold and Jost on existence of invariant tori which characterizes most integrable Hamiltonian systems – the modern version of the epicycles of Greek Astronomy.

Chapter 4 is a complement and a counterpart of the previous one. On the one hand it illustrates the Kronecker flow on tori that characterizes integrable systems; on the other hand it discusses in detail the theorem of Poincaré on nonexistence of first integrals which represents a first breach on the dream of integrability.

Chapter 5 deals with the dynamics in a neighbourhood of an equilibrium, paying attention to the problem of stability. The purpose is to create a bridge between the first half and the second half of the twentieth century. In view of the didactical purposes the first section is devoted to a discussion of the methods of solution of linear systems of differential equations, revisiting the classification of Poincaré. A detailed discussion of the Hamiltonian case is included. The rest of the chapter is concerned with the dynamics of a nonlinear system around an elliptic equilibrium. Its content is both theoretical, with a discussion of the existence of formal first integrals, and numerical, including both the methods based on the Poincaré section and the methods of algebraic manipulation. The last section introduces some analytical estimates, which in this case are not particularly difficult but open the way towards the strong results of the second half of the twentieth century.

Chapter 6 is a sort of technical *intermezzo*. It provides an introduction to the methods of Lie series and Lie transform that will be used in the next chapters. In particular it provides the basic analytical tools which allow us to prove the two main theorems of the next chapter. The formal methods discussed in this chapter have been developed starting around 1950, mainly in the milieu of Celestial Mechanics.

The third part begins with Chapter 7 and extends to the end of the notes. Most of its content is concerned with research developed in the second half of the twentieth century, with a remarkable exception: the homoclinic orbits discovered by Poincaré.

Chapter 7 introduces the methods of normal form for perturbed, close-to-integrable systems – the *general problem of dynamics*, as it has been qualified by Poincaré. The purpose is to provide an exposition of the methods developed by Poincaré and Birkhoff, and to discuss the long-standing problem of resonances and of small divisors. It also includes some comments concerning the way out from the difficulties raised by the theorem of Poincaré which will be fully developed in the next chapters.

Chapter 8 is devoted to the proof of the celebrated theorem of Kolmogorov on persistence of invariant tori. The first part of the chapter illustrates the concept of the Kolmogorov normal form, together with the fast convergence method which allowed him to solve a two-century-old problem. The proof of Kolmogorov is reproduced in complete form, only replacing the traditional method of generating functions in mixed variables with the method of Lie series. The second part provides a proof in a classical style, avoiding the fast convergence method, which hopefully sheds light on the behaviour of small divisors.

Chapter 9 deals with a further celebrated result: the exponential stability of Nekhoroshev. The theorem may be seen as a complement of Kolmogorov's one: it assures long-time stability for orbits in an open set, not just on strongly nonresonant invariant tori. The price to be paid is that stability is assured only for finite times, but hopefully is very long. The exposition is very technical but provides a complete proof.

Chapter 10, the final one, represents an attempt – a partial one – to illustrate how chaos and order may coexist in a perturbed Hamiltonian system.

The exposition is less technical with respect to the two previous chapters. It contains two basic pieces of information. The first one is an enhancement of the exponential stability of Nekhoroshev, showing that Kolmogorov's invariant tori are superexponentially stable. The second one is a reproposition of the last chapter of *Méthodes Nouvelles* of Poincaré, with the discovery of the phenomenon of homoclinic intersection and of chaotic behaviour.

The last chapter, Chapter 10, is intended as an encouragement to the reader who wants to keep exploring the Dynamics, thus filling the many holes that are left in the present notes and making a step forward. Therefore, it is not merely the end of the book: my hope is it will represent the beginning of a new journey. I will consider this as the best success of my work.

Note: Proofs of theorems, propositions, lemmas and corollaries are terminated by *'Q.E.D.'*. Examples are terminated by *'E.D.'*. Exercises are terminated by *'A.E.L.'*.

Expressions of Gratitude

Some years ago – it was between 1972 and 1973 – I was a student of Physics at the University of Milan. There I happened to meet my mentor Luigi Galgani – since then my good friend, whom I consider my scientific father. He was interested in the dynamics of the FPU system and in a work by P. Bocchieri, A. Scotti, B. Bearzi and A. Loinger on that subject, published a couple of years before. Coming back from a meeting in Cagliari, where Luigi Galgani and Antonio Scotti had met G. Contopoulos and learned from him about his methods of construction of a third integral, Luigi told a few people about Contopoulos's ideas. I was among these people, due to my dawning interest in the FPU problem; I went away so impressed that eventually I decided to begin my thesis work by studying the method of Contopoulos. A few months later, still during my thesis work, I was lucky to meet Contopoulos personally. My encounter with Luigi Galgani and George Contopoulos marked the beginning of my career; they are the first people to whom I must express my deep gratitude. This book represents the evolution of that lucky event, a bifurcation point in my life.

During the first years of my career I had a long and fruitful collaboration with Giancarlo Benettin and Jean Marie Strelcyn, together with Luigi. Our common work on Kolmogorov's and Nekhoroshev's theorems lies at the very basis of a significant part of the present notes. In more recent years I cultivated my interest in Celestial Mechanics. In that field I met three excellent students (among some others): Alessandro Morbidelli, Ugo Locatelli and Marco Sansottera. The present notes include also work that was developed thanks to their valuable collaboration.

I could add a long list of people I've met over the years, but it would be too long. I say that I met many people in a broad sense: an encounter may be direct – seminars, meetings and reciprocal visits are excellent opportunities to discuss with many people and to create a collaboration – or indirect – old books may establish a connection with scientists of the past and are a valuable source of knowledge. Some of them are mentioned in the references at the end of the book.

Now a few words concerning the preparation of this book. A few years ago Jürg Fröhlich told me, 'You should write a book.' I had written a considerable part of my notes at that time, making them available to students, but I constantly refrained from publishing them as a book despite the advice of friends and colleagues. Eventually I took Fröhlich's advice seriously: it was a good reason for undertaking such a job.

As I said at the beginning of my apology, the book is a reordered and partially rewritten collection of notes accumulated over many years. I have profited from the remarks, sometimes the criticism, of many students. I can not remember all their names, hence I do not name them, not to be unfair to anyone; but if my notes have been improved over time, this is due also to all of them.

Ugo Locatelli, Marco Sansottera, Simone Paleari, Giancarlo Benettin, Giuseppe Pucacco and Giuseppe Gaeta have taken the burden of reading at least part of the notes; their remarks and suggestions have been extremely useful.

Finally, I am grateful to my family: Anna, my wife; Cristina and Elena, my daughters; Elisa, Sergio and Pietro, my grandchildren. Perhaps they had no direct influence on the contents of this book, but they have helped create the family atmosphere and have endured my frequent moments of inattention when I was focused on writing. Without their patience I would never have finished this job.

In the run-up to the conclusion of the editorial process for this book, I would like to thank the editor Kaitlin Leach, at Cambridge University Press, for her courteous and valuable assistance.