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1
Hamiltonian Formalism

A major role in the development of the theory of classical dynamical systems is played by the Hamiltonian formulation of the equations of dynamics. This chapter is intended to provide a basic knowledge of the Hamiltonian formalism, assuming that the Lagrangian formalism is known. A reader already familiar with the canonical formalism may want to skip the present chapter.

The canonical equations were first written by Giuseppe Luigi Lagrangia (best known by the French version of his name, Joseph Louis Lagrange) as the last improvement of his theory of secular motions of the planets [140]. The complete form, later developed in what we now call Hamiltonian formalism, is due to William Rowan Hamilton [106][107][108]. A short sketch concerning the anticipations of Hamilton's work can be found in the treatise of Edmund Taylor Whittaker [209], §109.

In view of the didactical purpose of the present notes, the exposition in this chapter follows the traditional lines. The chapter includes some basic tools: the algebra of Poisson brackets and the elementary integration methods. Many examples are also included in order to illustrate how to write the Hamiltonian function for some models, often investigated using Newton's or Lagrange's equations.

1.1 Phase Space and Hamilton's Equations

The dynamical state of a system with n degrees of freedom is identified with a point on a $2n$-dimensional differentiable manifold, denoted by $\mathscr{F}$, endowed with canonically conjugated coordinates $(q,p) \equiv (q_1,\ldots,q_n,p_1,\ldots,p_n)$. The object of investigation is the evolution of the state of the system. The manifold $\mathscr{F}$ was named a *phase space* by Josiah Willard Gibbs [75].

The evolution of the system is determined by a real-valued Hamiltonian function $H\colon \mathscr{F} \to \mathbb{R}$ through the vector field defined by *Hamilton's equations* (also called *canonical equations*),

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$$\dot{q}_j = \frac{\partial H}{\partial p_j} \ , \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \ , \quad j = 1, \dots, n \ , \tag{1.1}$$

where $H = H(q,p)$. The Hamiltonian will be assumed to be a smooth (differentiable) function. In most examples the cases of $C^\infty(\mathscr{F},\mathbb{R})$ or even $C^\omega(\mathscr{F},\mathbb{R})$ Hamiltonians will be considered.

An *orbit* of the system is a smooth curve $\big(q(t),p(t)\big)$, for t in some (possibly infinite) interval, which is a solution of the canonical equations (1.1); the initial condition is written $q(0) = q_0$, $p(0) = p_0$. As a general fact, the solutions of a system of differential equations may be interpreted as a flow in phase space which transports every point (q_0,p_0) to another point $\big(q(t),p(t)\big) = \phi^t(q_0,p_0)$ after a time interval t, thus representing in some sense the dynamics in phase space as the motion of a fluid; the symbol ϕ^t represents the action of the flow. The corresponding orbit is thus represented as $\Omega(q_0,p_0) = \bigcup_t \phi^t(q_0,p_0)$, the union being made over the (possibly infinite) time interval of existence of the solution.

1.1.1 Autonomous versus Non-autonomous Systems

In most treatises the general case of a time-dependent Hamiltonian $H(q,p,t)$ is considered. It is customary to call such a system *non-autonomous*, in contrast with the time-independent case that is called *autonomous*.[1]

As a matter of fact, the non-autonomous case can be reduced to the autonomous one by a standard technique, already used by Henri Poincaré ([190], vol. I, § 12). The suggestion is to introduce the *extended phase space* by adding one more pair of canonically conjugated coordinates q_+, p_+, thus increasing the dimension of the phase space by 2; the special role played by the new variables will be emphasized by the special notation (q,q_+,p,p_+), although all canonical pairs should be considered on the same footing. Having given the Hamiltonian $H(q,p,t)$, let us introduce a new Hamiltonian

$$\tilde{H}\big(q,q_+,p,p_+\big) = H(q,p,q_+) + p_+ \ . \tag{1.2}$$

[1] The non-autonomous case typically arises when one considers a small system subjected to the time-dependent action of some external device, the state of which is not affected by the small part one is interested in. Examples are the restricted problem of three bodies, where the motion of a small body (e.g., an asteroid or a spacecraft) under the action of two big bodies moving on Keplerian orbits (e.g., Sun–Jupiter or Earth–Moon) is investigated; an electric charge acted on by an electromagnetic wave; a particle in an accelerator; &c. The autonomous case arises when an isolated system is considered.

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Thus, one has to consider the extended set of canonical equations

$$\begin{aligned} \dot q_j &= \frac{\partial H}{\partial p_j}\ , \quad \dot p_j = -\frac{\partial H}{\partial q_j}\ , \quad j = 1, \dots, n\ , \\ \dot q_+ &= 1 \quad\ , \quad \dot p_+ = -\frac{\partial H}{\partial q_+}\ . \end{aligned} \tag{1.3}$$

The third equation has the trivial solution $q_+(t) = t - t_0$, where t_0 is the initial time. In view of this, and having fixed the initial time t_0, the first two equations actually coincide with (1.1). Suppose now that we know a solution $q(t), p(t)$ of (1.1); if we replace it in the equation for $\dot p_+$, the r.h.s. turns out to be a known function of t only, so that the equation can be solved by a quadrature.[2] Therefore, the autonomous system (1.1) and the non-autonomous one (1.3) are fully equivalent.

Extending the phase space in order to transform a non-autonomous system into an autonomous one may appear to be an unnecessary complication: we have one more variable to take care of, although in a straightforward manner. However, this will offer us the opportunity of developing most of the theory in the framework of the autonomous systems, which makes the exposition definitely simpler.

1.1.2 Connection with the Lagrangian Formalism

It is traditional to introduce the Hamiltonian formalism starting from the equations of Lagrange.[3] The Hamiltonian function is introduced as the Legendre transform of the Lagrangian. One considers an n-dimensional differentiable manifold (the *configuration space*) endowed with (local) coordinates $q_1, \dots, q_n$, and its tangent space described by the generalized components $\dot q_1, \dots, \dot q_n$ of the velocities. The dynamical state of the system

[2] Actually no quadrature is needed, for an autonomous Hamiltonian $H(q,p)$ is a first integral, i.e., along the orbit $q(t), p(t)$ one has $H(q(t), p(t)) = E$, where $E = H(q(0), p(0))$ is the initial value – in most mechanical models it is the energy of the system. Since the Hamiltonian $\tilde H(q, q_+, p, p_+)$ in (1.2) is seen as an autonomous one we get the first integral

$$\tilde H(q(t), t - t_0, p(t), p_+(t)) = H(q(t), p(t), t) + p_+(t) = C\ .$$

The dynamics does not depend on the value of the constant C; hence we may well restrict the study to the manifold $\tilde H = 0$; this gives $p_+(t) = -H(q(t), p(t), t)$. Thus, the canonical momentum p_+ conjugated to the time t is identified with the energy E, which depends on time.

[3] For a deduction of the Lagrangian form of the equations of a mechanical system from Newton's equations see, for instance, [209], § 26. The original formulation can be found in Lagrange's treatise [139] and [143]. For a deduction of Hamilton's equations see [209], § 109 or [144], ch. VI.

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at a given time t is completely determined by the knowledge of $q(t), \dot{q}(t)$. The dynamics is determined by the Lagrangian function $L(q, \dot{q}, t)$ through the n differential equations of second order

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \ , \quad 1 \le j \le n \ . \tag{1.4}$$

In order to give the equations the Hamiltonian form, one introduces the momenta $p_1, \ldots, p_n$ conjugated to $q_1, \ldots, q_n$ defined as

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \ , \quad 1 \le j \le n \ ; \tag{1.5}$$

thus, the momenta are given as functions of q, $\dot{q}$ and t. The latter equations together with

$$\dot{p}_j = \frac{\partial L}{\partial q_j} \ , \quad 1 \le j \le n \tag{1.6}$$

are equivalent to the equations of Lagrange (1.4). If the condition

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_k}\right) \neq 0$$

is fulfilled, then (1.5) can be solved with respect to $\dot{q}_1, \ldots, \dot{q}_n$, thus giving $\dot{q}_j = \dot{q}_j(q, p, t)$, and the momenta can be used in place of the velocities $\dot{q}$ in order to determine the dynamical state.

The Hamiltonian function is defined as

$$H(q, p, t) = \sum_{j=1}^{n} p_j \dot{q}_j - L(q, \dot{q}, t)\Big|_{\dot{q}=\dot{q}(q,p,t)} \ , \tag{1.7}$$

where $\dot{q}$ must be replaced everywhere with its expression as a function of q, p, t. With the latter function the dynamics is determined by the canonical equations (1.1). The latter claim is proved as follows. Differentiate the function H as depending on the $3n+1$ variables q, $\dot{q}$, p and t, and then substitute the functions $\dot{q}(q, p, t)$, thus getting

$$\begin{aligned} dH &= \sum_{j=1}^{n} \left(\dot{q}_j \, dp_j + p_j \, d\dot{q}_j - \frac{\partial L}{\partial \dot{q}_j} \, d\dot{q}_j - \frac{\partial L}{\partial q_j} \, dq_j \right) - \frac{\partial L}{\partial t} \, dt \\ &= \sum_{j=1}^{n} \left(\dot{q}_j \, dp_j - \frac{\partial L}{\partial q_j} \, dq_j \right) - \frac{\partial L}{\partial t} \, dt \ . \end{aligned}$$

Here the definition, (1.5), of the momenta p has been used. The latter expression must be compared with the differential of H as a function of q, p, t, namely

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$$dH = \sum_{j=1}^{n} \left(\frac{\partial H}{\partial p_j}\, dp_j + \frac{\partial H}{\partial q_j}\, dq_j \right) + \frac{\partial H}{\partial t}\, dt \ .$$

By comparison of coefficients, also recalling (1.6), we get

$$\frac{\partial H}{\partial p_j} = \dot q_j \ , \quad \frac{\partial H}{\partial q_j} = -\frac{\partial L}{\partial q_j} = -\dot p_j \ , \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \ ,$$

namely the equations of Hamilton plus a relation between the time derivatives of H and L.

Example 1.1: *Free particle.* Let a point of mass m move in space under no forces. Using rectangular coordinates, the configuration space is identified with $\mathbb{R}^3$. The Lagrangian function coincides with the kinetic energy and reads

$$L = \frac{1}{2} m(\dot x^2 + \dot y^2 + \dot z^2) \ .$$

The momenta conjugated to the coordinates are $p_x = m\dot x, p_y = m\dot y, p_z = m\dot z$; they coincide with the components of the momentum of the particle. The phase space in this case is naturally identified with $\mathbb{R}^6$, and the Hamiltonian coincides with the kinetic energy, being

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) \ . \tag{1.8}$$

If the particle is acted on by a force depending on a potential $V(x, y, z)$, then the Lagrangian and the Hamiltonian functions are, respectively,

$$\begin{aligned} L &= \frac{1}{2} m(\dot x^2 + \dot y^2 + \dot z^2) - V(x, y, z) \ , \\ H &= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z) \ . \end{aligned} \tag{1.9}$$

In particular, the Hamiltonian represents the energy of the particle, the kinetic energy having been expressed in terms of the momenta. *E.D.*

Example 1.2: *One-dimensional harmonic oscillator.* Let now a point of mass m move on a straight line under the action of a perfectly elastic spring. The configuration space is $\mathbb{R}$, and the Lagrangian function is

$$L = \frac{1}{2} m\dot x^2 - \frac{1}{2} k x^2 \ .$$

Introducing the *angular frequency* $\omega = \sqrt{k/m}$, the Lagrangian changes to the following:[4]

$$L = \frac{1}{2}\dot x^2 - \frac{1}{2}\omega^2 x^2 \ .$$

[4] Recall that if the Lagrangian function is multiplied by an arbitrary non-zero factor, the equations of Lagrange do not change.

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The momentum conjugated to x is $p = \dot{x}$, so that the phase space is naturally identified with $\mathbb{R}^2$. The Hamiltonian is again the energy and is written as

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 x^2 \ . \tag{1.10}$$

E.D.

Example 1.3: *The pendulum.* A mass point is moving without friction on a vertical circle, so that it is subjected to gravity. The configuration space can be chosen to be a one-dimensional torus $\mathbb{T} = \mathbb{R}/(2\pi\,\mathbb{Z})$, namely the real line where the points x and $x + 2k\pi$ $(k \in \mathbb{Z})$ are identified. The resulting angular coordinate will be denoted by ϑ. The Lagrangian function may be written as

$$L = \frac{1}{2}\dot{\vartheta}^2 + \frac{g}{l}\cos\vartheta \ ,$$

where l is the length of the pendulum and g the constant acceleration due to local gravity. The momentum conjugated to ϑ is $p = \dot{\vartheta} \in \mathbb{R}$, and the phase space is naturally identified with $\mathbb{T} \times \mathbb{R}$. The Hamiltonian is

$$H = \frac{1}{2}p^2 - \frac{g}{l}\cos\vartheta \ . \tag{1.11}$$

E.D.

Example 1.4: *Motion under central forces.* The problem is to investigate the motion of a particle P of mass m acted on by a force in the direction PO, where O is a fixed point. Let us assume that the force field is spherically symmetric, so that there exists a potential $V(r)$, where r is the distance of P from the fixed point O. In view of the symmetry of the problem it is convenient to use spherical coordinates r, ϑ, φ, related to the rectangular coordinates x, y, z by

$$x = r\sin\vartheta\cos\varphi \ , \quad y = r\sin\vartheta\sin\varphi \ , \quad z = r\cos\vartheta \ .$$

Thus, the configuration space is $\mathbb{R}^+ \times (0,\pi) \times \mathbb{T}$ and the Lagrangian function reads

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\dot{\varphi}^2\sin^2\vartheta\right) - V(r) \ .$$

The momenta conjugated to the coordinates are $p_r = m\dot{r}$, $p_\vartheta = mr^2\dot{\vartheta}$ and $p_\varphi = mr^2\dot{\varphi}\sin^2\vartheta$. The Hamiltonian is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\vartheta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\vartheta}\right) + V(r) \ , \tag{1.12}$$

on the phase space $\mathbb{R}^+ \times (0,\pi) \times \mathbb{T} \times \mathbb{R}^3$. *E.D.*

Exercise 1.5: In galactic dynamics one often assumes the galaxy to be composed of a central nucleus with a large number of stars, and many other stars distributed in some way close to a plane orthogonal to the axis of rotation of the galaxy. The simplest approach consists in assuming that the distribution of the stars possesses an axial symmetry. Thus, one is led to

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study the motion of a single star close to the galactic plane, far from the nucleus, as subjected to an average potential $V(r,z)$, independent of ϑ in view of the axial symmetry. Show that in cylindrical coordinates r, ϑ, z with

$$x = r\cos\vartheta \ , \quad y = r\sin\vartheta$$

the Hamiltonian reads

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\vartheta^2}{r^2} + p_z^2\right) + V(r,z) \ . \qquad A.E.L.$$

Example 1.6: *The problem of n bodies.* The problem is to investigate the motion of a system of n particles in space, with a two-body interaction due to some potential. Let $\mathbf{x}_1,\dots,\mathbf{x}_n$ be the positions of the n bodies in a rectangular frame, so that the configuration space can be identified with $\mathbb{R}^{3n}$, and let $m_1,\dots,m_n$ be the masses of the bodies. Then the Lagrangian function reads

$$L = \frac{1}{2}\sum_{j=1}^{n} m_j\dot{\mathbf{x}}_j^2 - V(\mathbf{x}_1,\dots,\mathbf{x}_n) \ ,$$

where $V(\mathbf{x}_1,\dots,\mathbf{x}_n)$ is the potential energy. The momenta conjugated to the coordinates are $\mathbf{p}_j = m_j\dot{\mathbf{x}}_j$, $1 \le j \le n$. Therefore, the phase space can be identified with $\mathbb{R}^{6n}$, and the Hamiltonian reads

$$H = \sum_{j=1}^{n} \frac{\mathbf{p}_j^2}{2m_j} + V(\mathbf{x}_1,\dots,\mathbf{x}_n) \ . \tag{1.13}$$

The cases $n = 2$ and $n = 3$ are particularly interesting. The former one, the *problem of two bodies*, is the most natural approximation for the motion of a planet revolving around the Sun when the law of gravitation is applied in its correct form (i.e., the gravitational action of the planet on the Sun is taken into account). The latter one is the celebrated *problem of three bodies*, which was at the origin of the discovery of chaos by Poincaré. *E.D.*

Example 1.7: *Particle in a rotating frame.* We consider again the problem of a particle of mass m moving in space, acted on by some potential, and investigate its motion with respect to a frame uniformly rotating around a fixed axis. Let ξ,η,ζ be the coordinates of the particle with respect to a fixed rectangular frame, with the ζ axis coinciding with the fixed axis of the rotating frame. Let now x,y,z be rectangular coordinates in the rotating frame, chosen so that the two frames have a common origin, and the axes ζ and z coincide. Therefore, the two sets of coordinates are related via

$$\xi = x\cos\omega t - y\sin\omega t \ , \quad \eta = x\sin\omega t + y\cos\omega t \ , \quad \zeta = z \ ,$$

ω being the angular velocity of the rotating frame with respect to the fixed one. By substitution of the latter transformation in the Lagrangian function for a point in a fixed frame (see Example 1.1) one has

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$$L = \frac{1}{2}m\left[(\dot{x}-\omega y)^2 + (\dot{y}+\omega x)^2 + \dot{z}^2\right] - V(x,y,z,t)\ ,$$

$V(x,y,z,t)$ being the potential energy, which may depend on time due to the change of coordinates. The momenta conjugated to the coordinates are

$$p_x = m(\dot{x}-\omega y)\ , \quad p_y = m(\dot{y}+\omega x)\ , \quad p_z = m\dot{z}\ .$$

The Hamiltonian reads

$$H = \frac{1}{2m}\left(p_x^2+p_y^2+p_z^2\right) + \omega\left(yp_x - xp_y\right) + V(x,y,z,t)\ . \tag{1.14}$$

E.D.

Example 1.8: *The circular restricted problem of three bodies.* A remarkable example, which plays a fundamental role in Celestial Mechanics, is the so-called *restricted problem of three bodies*: the particle (named the *planetoid*) is attracted by two much bigger masses (the *primaries*, e.g., Sun–Jupiter or Earth–Moon) moving on Keplerian orbits without being affected by the planetoid. This model is a fundamental one when dealing with the dynamics of small bodies (e.g., inner planets, asteroids or spacecraft). The simplest approach is to assume that the primaries revolve uniformly on a circular orbit around their common centre of mass. It is also common to choose the units as follows: (i) the unit of length is the distance between the primaries; (ii) the period of the primaries is $T = 2\pi$; (iii) the masses of the primaries are set to be μ and $1-\mu$ with $0 < \mu < 1$, so that the total mass coincides with the mass unit. By the way, in these units the gravitational constant G takes the value $G = 1$. The rotating system is chosen so that the origin coincides with the barycentre and the primaries are in fixed positions $1-\mu$ and μ on the x axis. Therefore the potential turns out to be independent of t, since

$$V(x,y,z) = -\frac{m(1-\mu)}{\sqrt{(x-\mu)^2+y^2+z^2}} - \frac{m\mu}{\sqrt{(x+1-\mu)^2+y^2+z^2}}\ .$$

We are also allowed to remove the factor m from the Lagrangian, which is tantamount to substituting $m = 1$. Therefore, the Hamiltonian reads

$$H(x,y,z,p_x,p_y,p_z) = \frac{1}{2}(p_x^2+p_y^2+p_z^2) - \omega x p_y + \omega y p_x - \frac{1-\mu}{\sqrt{(x-\mu)^2+y^2+z^2}} - \frac{\mu}{\sqrt{(x+1-\mu)^2+y^2+z^2}}\ .$$

E.D.

Example 1.9: *Forced oscillations.* A simple but remarkable example in Physics is a harmonic oscillator acted on by a (small) periodic forcing. It is usual to write immediately the equation as, for example,

$$\ddot{x} + \omega^2 x = \varepsilon\cos\nu t\ , \tag{1.15}$$

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where ω is the proper frequency of the oscillator and ν is the frequency of the forcing term, while ε is a real parameter. The equation may be derived from the Lagrangian function

$$L = \frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2 + \varepsilon x \cos \nu t \ .$$

The Hamiltonian is easily found to be

$$H(x,p,t) = \frac{1}{2}(p^2 + \omega^2 x^2) - \varepsilon x \cos \nu t \ ,$$

where $p = \dot{x}$ is the momentum. The reader will notice that the Hamiltonian has the generic form $H = T + V(x,t)$, where $T = \dot{x}^2/2$ is the kinetic energy and $V(x,t)$ is the potential energy. An immediate generalization is to replace the forcing term with a more general function $xf(t)$, where $f(t)$ is time-periodic (i.e., $f(t+T) = f(t)$ for all t and some fixed T, e.g., $T = 2\pi/\nu$ in the preceding case). One may also add multiple periods or even include a non-periodic dependence on time, when useful. *E.D.*

Example 1.10: *The forced pendulum.* A definitely more difficult problem arises when one considers a forced *nonlinear* oscillator. A widely studied model is the forced pendulum, as described by the equation (denoting again by ϑ the coordinate)

$$\ddot{\vartheta} + \sin \vartheta = \varepsilon \cos \nu t \ , \tag{1.16}$$

where the forcing term may be replaced by a generic periodic function. The equation may be derived from the Lagrangian function

$$L = \frac{1}{2}\dot{\vartheta}^2 + \cos \vartheta + \varepsilon \vartheta \cos \nu t \ .$$

The Hamiltonian is easily found to be

$$H(\vartheta,p,t) = \frac{1}{2}p^2 - \cos \vartheta - \varepsilon \vartheta \cos \nu t \ .$$

A slightly different form of the Hamiltonian that is often used is

$$H_\varepsilon(\vartheta,p) = \frac{p^2}{2} - \cos \vartheta - \varepsilon \cos(\vartheta - t) \ , \quad \vartheta \in \mathbb{T} \ , \ p \in \mathbb{R} \ . \qquad E.D.$$

Example 1.11: *Autonomous mechanical system.* The general form of the Lagrangian function for an autonomous mechanical system may be written as

$$L(q,\dot{q}) = T(q,\dot{q}) - V(q) \ , \quad T(q,\dot{q}) = \frac{1}{2}\sum_{j,k=1}^{n} g_{j,k}(q)\dot{q}_j\dot{q}_k \ ,$$

where $T(q,\dot{q})$ is the kinetic energy in a general coordinate system, defined through a symmetric non-degenerate matrix $g(q)$ with elements $g_{j,k}(q)$, and

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$V(q)$ is the potential energy, independent of the velocities. The relation between generalized velocities and momenta is

$$p_j = \sum_{k=1}^{n} g_{jk}(q)\,\dot q_k \ , \quad \dot q_k = \sum_{j=1}^{n} [g^{-1}]_{kj}(q) p_j \ ,$$

where we have denoted by $[g^{-1}]_{j,k}$ the elements of the inverse matrix $g^{-1}(q)$. Thus we get

$$H = \frac{1}{2}\sum_{j,k=1}^{n} [g^{-1}]_{j,k} p_j p_k + V(q) \ . \tag{1.17}$$

E.D.

1.1.3 Compact Notation for the Canonical Equations

The canonical equations may be given a more compact form by introducing the $2n$ column vector $\mathbf{z}$ and the $2n \times 2n$ skew symmetric matrix J as

$$\mathbf{z} = (q_1, \ldots, q_n, p_1, \ldots, p_n)^\top \ , \quad \mathsf{J} = \begin{pmatrix} 0 & \mathsf{I}_n \\ -\mathsf{I}_n & 0 \end{pmatrix} \ , \tag{1.18}$$

where I_n is the $n \times n$ identity matrix. Remark that $\mathsf{J}^2 = -\mathsf{I}_{2n}$. We shall also introduce the operator

$$\partial_z = \left(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{2n}} \right)^\top \ . \tag{1.19}$$

With these notations the canonical equations take the compact form

$$\dot{\mathbf{z}} = \mathsf{J}\partial_z H \ . \tag{1.20}$$

The compact notation used here turns out to be useful in a few cases, since it allows us to write shorter expressions. Although most of the present notes are written in the traditional language, in some cases the compact notation will be adopted in order to simplify some calculations.

1.2 Dynamical Variables and First Integrals

A *dynamical variable* is a differentiable real function $f : \mathscr{F} \to \mathbb{R}$ with domain on the phase space (e.g., the kinetic energy, the potential energy and the Hamiltonian itself are dynamical variables, and so are the coordinates themselves). If the canonical coordinates evolve in time as $\big(q(t), p(t)\big)$, so does the dynamical variable $f\big(q(t), p(t)\big)$. In particular, if the flow $\big(q(t), p(t)\big)$ is determined by the canonical equations, (1.1), then the time evolution of $f(p, q)$ obeys

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