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Now in its second edition, this volume provides a uniquely detailed study of p-adic differential equations. Assuming only a graduate-level background in number theory, the text builds the theory from first principles all the way to the frontiers of current research, highlighting analogies and links with the classical theory of ordinary differential equations. The author includes many original results which play a key role in the study of p-adic geometry, crystalline cohomology, p-adic Hodge theory, perfectoid spaces, and algorithms for L-functions of arithmetic varieties. This updated edition contains five new chapters, which revisit the theory of convergence of solutions of p-adic differential equations from a more global viewpoint, introducing the Berkovich analytification of the projective line, defining convergence polygons as functions on the projective line, and deriving a global index theorem in terms of the Laplacian of the convergence polygon.

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p-adic Differential Equations

Second Edition

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CAMBRIDGE
UNIVERSITY PRESS

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University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

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www.cambridge.org
Information on this title: www.cambridge.org/9781009123341
DOI: 10.1017/9781009127684

First published 2010
Second edition 2022

A catalogue record for this publication is available from the British Library.

ISBN 978-1-009-12334-1 Hardback

Contents

Preface

This book is an outgrowth of a course taught by the author at MIT during fall 2007, on the subject of p-adic ordinary differential equations. The target audience was graduate students with some prior background in algebraic number theory, including exposure to p-adic numbers, but not necessarily any background in p-adic analytic geometry (of either the Tate or Berkovich flavors). The second edition was prepared during the 2020–2021 academic year.

Custom would dictate that this preface would continue with an explanation of what p-adic differential equations are and why they matter. Since we have included a whole chapter on this topic (Chapter 0), we instead devote this preface to a discussion of the origin of the book, its general structure, and what makes it different from previous books on the topic (including the first edition).

What was new in the first edition

The topic of p-adic differential equations has been treated in several previous books. Two that we used in preparing the MIT course, and to which we make frequent reference in the text, are those of Dwork, Gerotto, and Sullivan [149] and of Christol [89]. Another existing book is that of Dwork [145], but it is not a general treatise; rather, it focuses in detail on hypergeometric functions.

However, this book develops the theory of p-adic differential equations in a manner that differs significantly from most prior literature. The key differences include the following.

- We limit our use of cyclic vectors. This requires an initial investment in the study of matrix inequalities (Chapter 4) and lattice approximation arguments (especially Lemma 8.6.1), but pays off in the form of significantly stronger results.

- We introduce the notion of a Frobenius descendant (Chapter 10). This provides a complement to the older construction of Frobenius antecedents, particularly in dealing with certain boundary cases where the antecedent method does not apply.

As a result, we end up with some improvements of existing results, including the following (some of which can also be found in Christol's unpublished manuscript [93]):

- We refine the Frobenius antecedent theorem of Christol and Dwork (Theorem 10.4.2).
- We extend some results of Christol and Dwork, on the variation of the generic radius of convergence, to subsidiary radii (Theorem 11.3.2).
- We upgrade Young's geometric interpretation of subsidiary generic radii of convergence beyond the range of applicability of Newton polygons (Theorem 11.9.2).
- We quantify the Christol–Mebkhout decomposition theorem for differential modules on an annulus in ways which are applicable even when the modules are not solvable at a boundary (Theorems 12.2.2 and 12.3.1).
- We simplify the treatment of the theory of p-adic exponents (Theorems 13.5.5, 13.5.6, and 13.6.1).
- We sharpen the bound in the Christol transfer theorem to a disc containing a regular singularity with exponents in $\mathbb{Z}_p$ (Theorem 13.7.1).
- We generalize the Dieudonné–Manin classification theorem to difference modules over a complete nonarchimedean field (Theorem 14.6.3).
- We improve upon the Christol–Dwork–Robba effective bounds for solutions of p-adic differential equations (Theorem 18.2.1, Theorem 18.5.2) and some related bounds that apply in the presence of a Frobenius structure (Theorem 18.3.3). The latter can be used to recover a theorem of Chiarellotto and Tsuzuki concerning logarithmic growth of solutions of differential equations with Frobenius structure (Theorem 18.4.7).
- We state a relative version of the p-adic local monodromy theorem, formerly Crew's conjecture (Theorem 20.1.4). We describe in detail how it may be derived either from the p-adic index theory of Christol–Mebkhout, which we treat in detail in Chapter 13, or from the slope theory for Frobenius modules of Kedlaya, which we only sketch in Chapter 16.

Some of the new results are relevant in theory (in the study of higher-dimensional p-adic differential equations, largely in the context of the *semistable reduction problem* for overconvergent F-isocrystals, for which see [249] and [248]) or in practice (in the explicit computation of solutions of p-adic differential equations, e.g., for machine computation of zeta functions of particular varieties, for which see [244]). There is also some relevance entirely outside of number theory, to the study of flat connections on complex analytic varieties (see [250]).

Although some of the intended applications involve higher-dimensional p-adic analytic spaces, this book treats exclusively p-adic *ordinary* differential equations. In joint work with Liang Xiao [271], we have developed some extensions to higher-dimensional spaces.

What's new in the second edition

The second edition incorporates corrections to various errors in the original text. It also includes some new material, largely concerning developments that emerged after the publication of the first edition. We single out a few highlights:

- Many old and new references have been added to the chapter notes.
- In Chapter 7, we have added a discussion of the index of a meromorphic differential module (Section 7.4) and an example of the Stokes phenomenon (Section 7.6).
- In Chapter 12, we have expanded the discussion of clean modules and spectral decompositions (Section 12.8).
- In Chapter 13, we have eliminated the forward reference from Lemma 13.5.4 to Corollary 18.2.5 by providing an alternate proof of the former, corrected the proof of Theorem 13.6.1, and added a discussion of Liouville partitions (Section 13.8).
- In Chapter 18, we have expanded the discussion of logarithmic growth to include results of André and Ohkubo (Section 18.4), and corrected the statement and proof of Theorem 18.5.2.
- We have created a new Part VI consisting of Chapters 20 and 21 plus one new chapter (Chapter 22).
- In Chapter 20, we have added a theorem of Tsuzuki on minimal slope quotients (Theorem 20.6.7).
- We have added Chapter 22 to present a form of the p-adic local monodromy theorem that holds in the absence of a Frobenius structure. This includes the definition and analysis of modules of cyclic type (Section 22.2).
- Chapters 22, 23, and 24 from the first edition appear as Appendices A, B, and C in this edition.
- Part VII is entirely new to the second edition. See below.

Structure of the book

Each individual chapter of this book exhibits the following basic structure. Before the body of the chapter, we provide a brief introduction explaining what is to be discussed and often setting some running notations or hypotheses. After the body of the chapter, we include a section of afternotes, in which we provide detailed references for results in that chapter, fill in historical details, and add additional comments. (This practice is modeled on [172], although we cannot pretend to the level of detail achieved therein.) Note that we have a habit of attributing to various authors slightly stronger versions of their theorems than the ones they originally stated; to avoid complicating the discussion in the text, we resolve these misattributions in the afternotes instead. (See also the thematic bibliography of [259] for additional references, albeit without much context.) At the end of the chapter, we typically include a few exercises; a fair number of these request proofs of results which are stated and used in the text, but whose proofs pose no unusual difficulties.

The chapters themselves are grouped into several parts, which we now describe briefly. (Chapter 0, being introductory, does not fit into this grouping.)

Part I is preliminary, collecting some basic tools of p-adic analysis. However, it also includes some facts of matrix analysis (the variation of numerical invariants attached to matrices as a function of the matrix entries) which may not be familiar to the typical reader.

Part II introduces some formalism of differential algebra, such as differential rings and modules, twisted polynomials, and cyclic vectors, and applies these to fields equipped with a nonarchimedean norm.

Part III begins the study of p-adic differential equations in earnest, developing some basic theory for differential modules on rings and annuli, including the Christol–Dwork theory of variation of the generic radius of convergence, and the Christol–Mebkhout decomposition theory. We also include a treatment of p-adic exponents, culminating in the Christol–Mebkhout structure theorem for p-adic differential modules on an annulus satisfying the Robba condition (i.e., having intrinsic generic radius of convergence everywhere equal to 1).

Part IV introduces some formalism of difference algebra and presents (without full proofs) the theory of slope filtrations for Frobenius modules over the Robba ring.

Part V introduces the concept of a Frobenius structure on a p-adic differential module. We also discuss effective convergence bounds for solutions of p-adic differential equations.

Part VI presents the p-adic local monodromy theorem (formerly Crew's conjecture) and the proof techniques using either p-adic exponents or Frobenius slope filtrations. We also introduce a new approach that gives a version of the theorem that applies in the absence of a Frobenius structure.

Part VII revisits the theory of convergence of solutions of p-adic differential equations from a more global viewpoint. We introduce the Berkovich analytification of the projective line, define and study convergence polygons as functions on the projective line, and state a global index theorem in terms of the Laplacian of the convergence polygon.

The appendices consist of a series of brief discussions of several areas of application of the theory of p-adic differential equations. While they are formatted like chapters (without exercises), their textual style is somewhat more didactic and much less formal than in the main text (excluding Chapter 0); they are meant primarily as suggestions for further reading.

Prerequisites

As noted above, we have not assumed that the reader is familiar with rigid analytic geometry, and so have phrased all statements more concretely in terms of rings and modules. Although we expect that the typical reader has at least a passing familiarity with p-adic numbers (e.g., at the level of Gouvêa's text [178]), for completeness we begin with a rapid development of the algebra of complete rings and fields. This development, when read on its own, may appear somewhat idiosyncratic; its design is justified by the reuse of some material in later chapters.

We would ultimately like to think that the background needed is that of a two-semester undergraduate abstract algebra sequence. However, this may be a bit too optimistic; some basic notions from commutative algebra do occasionally intervene, including flat modules, exact sequences, and the snake lemma. It may be helpful to have a well-indexed text within arm's reach; we like Eisenbud's book [154], but the far slimmer book by Atiyah and Macdonald [24] should also suffice. (At the opposite extreme, we are also partial to the massive Stacks Project [378].)

Leitfaden

Figure 0.1 indicates logical dependencies among the chapters, with each part of the book represented in a single row. To keep the diagram manageable, we grouped together some chapters (1–3 and 9–12), and omitted Chapter 0 and the appendices.

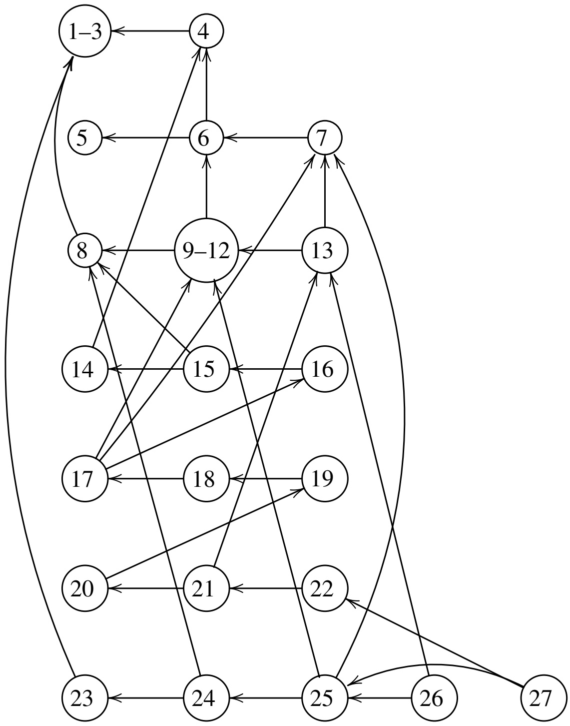

Figure 0.1 Diagram of logical dependencies among chapters

Acknowledgments

We thank the participants of the MIT course 18.787 (Topics in Number Theory, fall 2007) for numerous comments on the lecture notes which ultimately became this book. Particular thanks are due to Ben Brubaker and David Speyer for giving guest lectures, and to Chris Davis, Hansheng Diao, David Harvey, Raju Krishnamoorthy, Ruochuan Liu, Eric Rosen, and especially Liang Xiao for providing feedback. Additional feedback on drafts of the first edition was provided by Francesco Baldassarri, Laurent Berger, Bruno Chiarellotto, Gilles Christol, Ricardo García López, Tim Gowers, and Andrea Pulita. Feedback on the published first edition was provided by Francesco Baldassarri, Joshua Ciappara, Michel Matignon, Grant Molnar, Takahiro Nakagawa, Shun Ohkubo, David Savitt, Atsushi Shiho, Junecue Suh, Peiduo Wang, and Shuyang Ye.

During the preparation of the course and of the first edition, the author was supported by a National Science Foundation CAREER grant (DMS-0545904), a Sloan Research Fellowship, MIT's NEC Research Support Fund, and the MIT Cecil and Ida Green Career Development Professorship. During the preparation of the second edition, the author was supported by NSF (DMS-1802161, DMS-2053473) and the Stefan E. Warschawski Professorship at UC San Diego.