1 Introduction

by Andrew J. Blumberg, Teena Gerhardt, Michael A. Hill

1.1 Goals of this book

The modern era in homotopy theory began in the 1960s with the profound realization, first codified by Boardman in his construction of *the stable category*, that the category of spaces up to stable homotopy equivalence is equipped with a rich algebraic structure, formally similar to the derived category of a commutative ring R. For example, for pointed spaces the natural map from the categorical coproduct to the categorical product becomes more and more connected as the pieces themselves become more and more connected. In the limit, this map becomes a stable equivalence, just as finitely indexed direct sums and direct products coincide for R-modules.

From this perspective, the objects of the stable category are modules over an initial commutative ring object that replaces the integers: the sphere spectrum. However, technical difficulties immediately arose. Whereas the tensor product of R-modules is an easy and familiar construction, the analogous construction of a symmetric monoidal smash product on spectra seemed to involve a huge number of ad hoc choices [1]. As a consequence, the smash product was associative and commutative only up to homotopy. The lack of a good point-set symmetric monoidal product on spectra precluded making full use of the constructions from commutative algebra in this setting — even just defining good categories of modules over a commutative ring spectrum was difficult. In many ways, finding ways to rectify this and to make the guiding metaphor provided by "modules over the sphere spectrum" precise has shaped the last 60 years of homotopy theory.

This book arose from a desire by the editors to have a reference to give to their students who have taken a standard algebraic topology sequence and who want to learn about spectra and structured ring spectra. While there are many excellent texts which introduce students to the basic ideas of homotopy theory and to spectra, there has not been a place for students to engage directly with the ideas needed to connect with commutative ring spectra and work with these objects. This book strives to provide an introduction to this whole circle of ideas, describing the tools that homotopy theorists have developed to build, explore, and use symmetric monoidal categories of spectra that refine the stable homotopy category:

1. model category structures on symmetric monoidal categories of spectra,
2. stable ∞-categories, and
3. operads and operadic algebras.

These three concepts are closely intertwined, and they all engage deeply with a fundamental principle: if the choices for some construction or map are parameterized by a space, then recording that space as part of the data makes the construction more natural.

To make this maxim precise in practice, we must keep track of the spaces of maps between objects in our categories, not just sets of maps, describing the homotopies by which two equivalent maps are seen to be equivalent. A first example of this is given by the cup product on ordinary cohomology. Students in a first algebraic topology class learn that while the cochains on a space with coefficients in a commutative ring are not a commutative ring, the cohomology of a space is canonically a graded commutative ring. Steenrod observed that over $\mathbb{F}_p$, we can keep track of cochains that enforce the symmetry between $a \smile b$ and $b \smile a$, and out of these, we can build a hierarchy of cochains and, when $a = b$, cocycles in increasingly high degree: the Steenrod reduced powers [283]. May recast this via operads: mathematical objects which exactly record spaces parametrizing particular kinds of multiplications [198, 194].

Again returning to our maxim, we want to be sure that our constructions, including of the mapping spaces, are homotopically meaningful in the sense that the resulting homotopy type of any output depends only on the homotopy types of the inputs. Model categories provide one way to ensure this, giving us not only checkable conditions to facilitate computation but also a language and explanation for fundamental constructions in homological algebra like resolutions and derived functors. More recently, ∞-categories have given another way to ensure homotopically meaningful information by recording this data from the very beginning.

Homotopy theory is at an inflection point, with much of the older literature written in the language of model categories and with newer results and machinery expressed using ∞-categories. Both approaches have distinct benefits, and we provide an introduction to both: our aim is to give people learning about stable categories and structured ring spectra a way to connect with both “neoclassical” tools and newer ones.

The book closes with applications of the tools so developed, showing how the machinery of ∞-categories allows us to fully realize Boardman’s observation and “do algebraic geometry” with commutative ring spectra. Transformative work of Goerss–Hopkins–Miller in the last 1990s ushered in the era of spectral algebraic geometry, showing first that the Lubin–Tate deformation theory of formal groups naturally lifts to a diagram of commutative ring spectra and then that the structure sheaf of the moduli stack of formal groups has an essentially unique lift to a sheaf of commutative ring spectra [106, 126]. This produced a host of new cohomology theories which are naturally tied to universal constructions in algebraic geometry and moduli problems. Additionally, it refined classical invariants of rings like modular forms to invariants of ring spectra: topological modular forms. Lurie has created a vast generalization of this, showing how one can lift algebraic geometry whole-cloth to commutative ring spectra, creating spectral algebraic geometry. The final chapter of this book provides an introduction to this new area.

1.2 Summaries of the chapters

Chapter 2 (Riehl)

The chapter begins by framing a foundational question: What do we mean by the homotopy category of a category and by derived functors? It proceeds through a historical arc: describing first categories of fractions, then moving on to Quillen's theory of model categories and their simplicial enrichments, and finally describing the newer, $(\infty, 1)$-categories. The goal here is to introduce the reader to the basic tools that will be used, fitting them into a broader narrative, demonstrating how they can be used, and connecting everything clearly to the literature for further study.

Chapter 3 (Dugger)

This chapter gives a comprehensive overview of the modern symmetric monoidal categories of spectra that were invented in the 90s: symmetric spectra, orthogonal spectra, and EKMM spectra. The technical foundations are carried out in the setting of model categories, and there is an emphasis on concrete formulas for the smash product and related constructions. The goal is for the reader to become comfortable with working in these categories of spectra.

Chapter 4 (Barwick)

This chapter returns to the construction of the category of spectra and explains the approach to spectra and stable categories more generally in the framework of $(\infty, 1)$-categories. We hope that comparing and contrasting the treatment in this chapter and the preceding one will give a flavor of the similarities and differences between the two technical approaches for abstract homotopy theory. Of necessity, many details about the underlying foundations are left to the references, but enough detail is provided to indicate how the theory works.

Chapter 5 (Mandell)

This chapter is a thorough treatment of the theory of operadic algebras in modern homotopy theory. It gives a streamlined view of the foundations, collecting in one place results that are scattered throughout the literature, with a unifying viewpoint on techniques for understanding the homotopy theory of operadic algebras and modules.

Chapter 6 (Richter)

This chapter gives a broad sampling of applications of commutative ring spectra in modern stable homotopy theory. Beginning with a treatment of the foundations, it then surveys applications in topological Hochschild homology, obstruction theory and topological André–Quillen homology, and the Picard and Brauer groups.

Chapter 7 (Lawson)

This chapter gives a detailed introduction to the theory of Bousfield localization, starting from the basic constructions and studying the multiplicative properties of the localization in the context of structured ring spectra. Bousfield localization is one of the most important techniques in the modern arsenal, and the goal of this chapter is to prepare the reader to understand how to use it.

Chapter 8 (Rezk)

This chapter draws on all of the earlier sections, showing how the machinery developed allows us to "do algebraic geometry" in a very general context. The chapter begins discussing ∞-topoi and sheaves on them, providing along the way useful tools and ways to reinterpret results to show how these constructions can be used. It then moves into more algebraic geometry notions, exploring how classical notions like étale morphism, affine and projective spaces, and stacks lift to commutative ring spectra. This culminates in a treatment of Lurie's refinement of the Goerss–Hopkins–Miller theorem that the structure sheaf of the moduli stack of elliptic curves lifts to commutative ring spectra.

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More Information

2 Homotopical categories: from model categories to $(\infty, 1)$-categories

by Emily Riehl

2.1 The history of homotopical categories

A *homotopical category* is a category equipped with some collection of morphisms traditionally called "weak equivalences" that somewhat resemble isomorphisms but fail to be invertible in any reasonable sense, and might in fact not even be reversible: that is, the presence of a weak equivalence $X \xrightarrow{\sim} Y$ need not imply the presence of a weak equivalence $Y \xrightarrow{\sim} X$. Frequently, the weak equivalences are defined as the class of morphisms in a category K that are "inverted by a functor" $F \colon \mathsf{K} \to \mathsf{L}$, in the sense of being precisely those morphisms in K that are sent to isomorphisms in L. For instance:

- Weak homotopy equivalences of spaces or spectra are those maps inverted by the homotopy group functors $\pi_* \colon \mathsf{Top} \to \mathsf{GrSet}$ or $\pi_* \colon \mathsf{Spectra} \to \mathsf{GrAb}$.
- Quasi-isomorphisms of chain complexes are those maps inverted by the homology functor $H_* \colon \mathsf{Ch} \to \mathsf{GrAb}$.
- Equivariant weak homotopy equivalences of G-spaces are those maps inverted by the homotopy functors on the fixed point subspaces for each compact subgroup of G.

The term used to describe the equivalence class represented by a topological space up to weak homotopy equivalence is a *homotopy type*. Since the weak homotopy equivalence relation is created by the functor π_*, a homotopy type can loosely be thought of as a collection of algebraic invariants of the space X, as encoded by the homotopy groups $\pi_* X$. Homotopy types live in a category called the *homotopy category of spaces*, which is related to the classical category of spaces as follows: a genuine continuous function $X \to Y$ certainly represents a map (graded homomorphism) between homotopy types. But a weak homotopy equivalence of spaces, defining an isomorphism of homotopy types, should now be regarded as formally invertible.

In their 1967 manuscript *Calculus of fractions and homotopy theory*, Gabriel and Zisman [100] formalized the construction of what they call the *category of fractions* associated to any class of morphisms in any category together with an associated localization functor $\pi \colon \mathsf{K} \to \mathsf{K}[\mathcal{W}^{-1}]$ that is universal among functors with domain K that invert the class $\mathcal{W}$ of weak equivalences. This construction and its universal property are presented in §2.2. For instance, the homotopy category of spaces arises as the category of fractions associated to the weak homotopy equivalences of spaces.

There is another classical model of the homotopy category of spaces that defines an equivalent category. The objects in this category are the *CW-complexes*, spaces built by gluing disks along their boundary spheres, and the morphisms are now taken to be homotopy classes of maps. By construction the isomorphisms in this category are the homotopy equivalences of CW-complexes. Because any space is weak homotopy equivalent to a CW-complex and because Whitehead's theorem proves that the weak homotopy equivalences between CW-complexes are precisely the homotopy equivalences, it can be shown that this new homotopy category is equivalent to the Gabriel–Zisman category of fractions.

Quillen introduced a formal framework which draws attention to the essential features of these equivalent constructions. His axiomatization of an abstract "homotopy theory" was motivated by the following question: When does it make sense to invert a class of morphisms in a category and call the result a homotopy category, rather than simply a localization? In the introduction to his 1967 manuscript *Homotopical Algebra* [229], Quillen reports that Kan's theorem that the homotopy theory of simplicial groups is equivalent to the homotopy theory of connected pointed spaces [143] suggested to Quillen that simplicial objects over a suitable category A might form a homotopy theory analogous to classical homotopy theory in algebraic topology. In pursuing this analogy he observed that

> there were a large number of arguments which were formally similar to well-known ones in algebraic topology, so it was decided to define the notion of a homotopy theory in sufficient generality to cover in a uniform way the different homotopy theories encountered. [229, pp. 1–2]

Quillen named these homotopy theories *model categories*, meaning "categories of models for a homotopy theory." He entitled his explorations "homotopical algebra," as they describe both a generalization of and a close analogy to homological algebra — in which the relationship between an abelian category and its derived category parallels the relationship between a model category and its homotopy category. We introduce Quillen's model categories and his construction of their homotopy categories as a category of "homotopy" classes of maps between sufficiently "fat" objects in §2.3. A theorem of Quillen proven as Theorem 2.3.29 below shows that the weak equivalences in any model category are precisely those morphisms inverted by the Gabriel–Zisman localization functor to the homotopy category. In particular, in the homotopical categories that we will most frequently encounter, the weak equivalences satisfy a number of closure properties, to be introduced in Definition 2.3.1.

To a large extent, homological algebra is motivated by the problem of constructing derived versions of functors between categories of chain complexes that fail to preserve weak equivalences. A similar question arises in Quillen's model categories. Because natural transformations can point either to or from a given functor, derived functors come with a "handedness": either left or right. In §2.4, we introduce dual notions of left and right Quillen functors between model categories and construct their derived functors via a slightly unusual route that demands a stricter (but in our view improved) definition of derived functors than the conventional one. In parallel, we study the additional properties borne by Quillen's original model structure on simplicial sets, later axiomatized by Hovey [130] in the notion of a monoidal or enriched model category, which derives to define monoidal structures or enrichments on the homotopy category.

These considerations also permit us to describe when two "homotopy theories" are equivalent. For instance, the analogy between homological and homotopical algebra is solidified by a homotopical reinterpretation of the Dold–Kan theorem as an equivalence between the homotopy theory of simplicial objects of modules and chain complexes of modules presented in Theorem 2.4.33.

As an application of the theory of derived functors, in §2.5 we study homotopy limits and colimits, which correct for the defect that classically defined limit and colimit constructions frequently fail to be weak equivalence invariant. We begin by observing that the homotopy category admits few strict limits. It does admit weak ones, as we shall see in Theorem 2.5.3, but their construction requires higher homotopical information which will soon become a primary focus.

By convention, a full Quillen model structure can only be borne by a category possessing all limits and colimits, and hence the homotopy limits and homotopy colimits introduced in §2.5 are also guaranteed to exist. This supports the point of view that a model category is a presentation of a homotopy theory with all homotopy limits and homotopy colimits. In a series of papers from 1980 [89, 87, 88], Dwyer and Kan describe more general "homotopy theories" as *simplicial localizations* of categories with weak equivalences, which augment the Gabriel–Zisman category of fractions with homotopy types of the mapping spaces between any pair of objects. The *hammock localization* construction described in §2.6 is very intuitive, allowing us to reconceptualize the construction of the category of fractions not by imposing relations in the same dimension, but by adding maps in the next dimension — "imposing homotopy relations" if you will.

The hammock localization defines a simplicially enriched category associated to any homotopical category. A simplicially enriched category is a non-prototypical exemplification of the notion of an $(\infty, 1)$-*category*, that is, a category weakly enriched over ∞-groupoids or homotopy types. Model categories also equip each pair of their objects with a well-defined homotopy type of maps, and hence also present $(\infty, 1)$-categories. Before exploring $(\infty, 1)$-*categories* in a systematic way, in §2.7 we introduce the most popular model, the *quasi-categories* first defined in 1973 by Boardman and Vogt [48] and further developed by Joyal [140, 141] and Lurie [169].

In §2.8 we turn our attention to other models of $(\infty, 1)$-categories, studying six in total: quasi-categories, Segal categories, complete Segal spaces, naturally marked quasi-categories, simplicial categories, and relative categories. The last two models are strictly-defined objects, which are quite easy to define, but the model categories in which they live are poorly behaved. By contrast, the first four of these models live in model categories that have many pleasant properties, which are collected together in a new axiomatic notion of an ∞-*cosmos*.

After introducing this abstract definition, we see in §2.9 how the ∞-cosmos axiomatization allows us to develop the basic theory of these four models of $(\infty, 1)$-categories model-independently, that is, simultaneously and uniformly across these models. Specifically, we study adjunctions and equivalences between $(\infty, 1)$-categories and limits and colimits in an $(\infty, 1)$-category to provide points of comparison for the corresponding notions of Quillen adjunction, Quillen equivalence, and homotopy limits and colimits developed for model categories in §2.4 and §2.5. A brief epilogue, §2.10, contains a few closing thoughts and anticipates future chapters in this volume.

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2.2 Categories of fractions and localization

In one of the first textbook accounts of abstract homotopy theory [100], Gabriel and Zisman construct the universal category that inverts a collection of morphisms together with accompanying "calculi-of-fractions" techniques for calculating this categorical "localization." Gabriel and Zisman prove that a class of morphisms in a category with finite colimits admits a "calculus of left fractions" if and only if the corresponding localization preserves them, which then implies that the category of fractions also admits finite colimits [100, §1.3]; dual results relate finite limits to their "calculus of right fractions." For this reason, their calculi of fractions fail to exist in the examples of greatest interest to modern homotopy theorists, and so we will not introduce them here, focusing instead in §2.2.1 on the general construction of the category of fractions.

2.2.1 The Gabriel–Zisman category of fractions

For any class of morphisms $\mathcal{W}$ in a category K, the category of fractions $\mathsf{K}[\mathcal{W}^{-1}]$ is the universal category equipped with a functor $\iota\colon \mathsf{K} \to \mathsf{K}[\mathcal{W}^{-1}]$ that inverts $\mathcal{W}$, in the sense of sending each morphism to an isomorphism. Its objects are the same as the objects of K and its morphisms are finite zigzags of morphisms in K, with all "backwards" arrows finite composites of arrows belonging to $\mathcal{W}$, modulo a few relations which convert the canonical graph morphism $\iota\colon \mathsf{K} \to \mathsf{K}[\mathcal{W}^{-1}]$ into a functor and stipulate that the backwards copies of each arrow in $\mathcal{W}$ define two-sided inverses to the morphisms in $\mathcal{W}$.

Definition 2.2.1 (category of fractions [100, 1.1]). For any class of morphisms $\mathcal{W}$ in a category K, the **category of fractions** $\mathsf{K}[\mathcal{W}^{-1}]$ is a quotient of the free category on the directed graph obtained by adding backwards copies of the morphisms in $\mathcal{W}$ to the underlying graph of the category K modulo certain relations:

- Adjacent arrows pointing forwards can be composed.
- Forward-pointing identities may be removed.
- Adjacent pairs of zigzags

$$x \xrightarrow{s} y \xleftarrow{s} x \quad \text{or} \quad y \xleftarrow{s} x \xrightarrow{s} y$$

indexed by any $s \in \mathcal{W}$ can be removed.[1]

The image of the functor $\iota\colon \mathsf{K} \to \mathsf{K}[\mathcal{W}^{-1}]$ is comprised of those morphisms that can be represented by unary zigzags pointing forwards.

The following proposition expresses the 2-categorical universal property of the category of fractions construction in terms of categories $\mathsf{Fun}(\mathsf{K},\mathsf{M})$ of functors and natural transformations:

Proposition 2.2.2 (the universal property of localization [100, 1.2]). *For any category* M, *restriction along* ι *defines a fully faithful embedding*

$$\begin{array}{ccc} \mathsf{Fun}(\mathsf{K}[\mathcal{W}^{-1}],\mathsf{M}) & \xhookrightarrow{-\circ\iota} & \mathsf{Fun}(\mathsf{K},\mathsf{M}) \\ & \searrow^{\cong} \quad \nearrow & \\ & \underset{\mathcal{W}\mapsto\cong}{\mathsf{Fun}}(\mathsf{K},\mathsf{M}) & \end{array}$$

defining an isomorphism

$$\mathsf{Fun}(\mathsf{K}[\mathcal{W}^{-1}],\mathsf{M}) \cong \underset{\mathcal{W}\mapsto\cong}{\mathsf{Fun}}(\mathsf{K},\mathsf{M})$$

of categories onto its essential image, the full subcategory spanned by those functors that invert $\mathcal{W}$.

Proof. As in the analogous case of rings, the functor $\iota\colon \mathsf{K} \to \mathsf{K}[\mathcal{W}^{-1}]$ is an epimorphism and so any functor $F\colon \mathsf{K} \to \mathsf{M}$ admits at most one extension along ι. To show that any functor $F\colon \mathsf{K} \to \mathsf{M}$ that inverts $\mathcal{W}$ does extend to $\mathsf{K}[\mathcal{W}^{-1}]$, we define a graph morphism from the graph described in Definition 2.2.1 to M by sending the backwards copy of s to the isomorphism $(Fs)^{-1}$ and thus a functor from the free category generated by this graph to M. Functoriality of F ensures that the enumerated relations are respected by this functor, which therefore defines an extension $\hat{F}\colon \mathsf{K}[\mathcal{W}^{-1}] \to \mathsf{M}$ as claimed.

The 2-dimensional aspect of this universal property follows from the 1-dimensional one by considering functors valued in arrow categories [146, §3]. □

Example 2.2.3 (groupoid reflection). When all the morphisms in K are inverted, the universal property of Proposition 2.2.2 establishes an isomorphism $\mathsf{Fun}(\mathsf{K}[\mathsf{K}^{-1}],\mathsf{M}) \cong \mathsf{Fun}(\mathsf{K},\mathrm{core}\mathsf{M})$ between functors from the category of fractions of K to functors valued in the **groupoid core**, which is the maximal subgroupoid contained in M. In this

[1] It follows that adjacent arrows in $\mathcal{W}$ pointing backwards can also be composed whenever their composite in K also lies in $\mathcal{W}$.

way, the category of fractions construction specializes to define a left adjoint[2] to the inclusion of groupoids into categories:

$$\mathsf{Cat} \; \underset{\text{core}}{\overset{\text{fractions}}{\underset{\perp}{\overset{\perp}{\longleftarrow}}}} \; \mathsf{Gpd}$$

The universal property of Proposition 2.2.2 applies to the class of morphisms inverted by any functor admitting a fully faithful right adjoint [100, 1.3]. In this case, the category of fractions defines a *reflective subcategory* of K, which admits a variety of useful characterizations, one being as the *local* objects orthogonal to the class of morphisms being inverted [238, 4.5.12, 4.5.vii, 5.3.3, 5.3.i]. For instance, if $R \to R[S^{-1}]$ is the localization of a commutative ring at a multiplicatively closed set, then the category of $R[S^{-1}]$-modules defines a reflective subcategory of the category of R-modules [238, 4.5.14], and hence the extension of scalars functor $R[S^{-1}] \otimes_R -$ can be understood as a Gabriel–Zisman localization.

However, reflective subcategories inherit all limits and colimits present in the larger category [238, 4.5.15], which is not typical behavior for categories of fractions that are "homotopy categories" in a sense to be discussed in §5.8. With the question of when a category of fractions is a homotopy category in mind, we now turn our attention to Quillen's homotopical algebra.

2.3 Model category presentations of homotopical categories

A question that motivated Quillen's introduction of model categories [229] and also Dwyer, Kan, Hirschhorn, and Smith's later generalization [92] is: When is a category of fractions a homotopy category? Certainly, the localization functor must invert some class of morphisms that are suitably thought of as "weak equivalences." Perhaps these weak equivalences coincide with a more structured class of "homotopy equivalences" on a suitable subcategory of "fat" objects that spans each weak equivalence class — such as given in the classical case by Whitehead's theorem that any weak homotopy equivalence between CW complexes admits a homotopy inverse — in such a way that the homotopy category is equivalent to the category of homotopy classes of maps in this full subcategory. Finally, one might ask that the homotopy category admit certain derived constructions, such as the loop and suspension functors definable on the homotopy category of based spaces. On account of this final desideratum, we will impose the blanket requirement that a category that bears a model structure must be complete and cocomplete.

[2] More precisely, this left adjoint takes values in a larger universe of groupoids, since the category of fractions $\mathsf{K}[\mathsf{K}^{-1}]$ associated to a locally small category K need not be locally small. Toy examples illustrating this phenomenon are easy to describe. For instance, let K be a category with a proper class of objects whose morphisms define a "double asterisk": each non-identity morphism has a common domain object and for each other object there are precisely two non-identity morphisms with that codomain.