HIGHER SPECIAL FUNCTIONS

Higher special functions emerge from boundary eigenvalue problems of Fuchsian differential equations with more than three singularities.

This detailed reference provides solutions for singular boundary eigenvalue problems of linear ordinary differential equations of second order, exploring previously unknown methods for finding higher special functions.

Starting from the fact that it is the singularities of a differential equation that determine the local, as well as the global, behaviour of its solutions, the author develops methods that are both new and efficient and lead to functional relationships that were previously unknown.

All the developments discussed are placed within their historical context, allowing the reader to trace the roots of the theory back through the work of many generations of great mathematicians. Particular attention is given to the work of George Cecil Jaffé, who laid the foundation with the calculation of the quantum mechanical energy levels of the hydrogen molecule ion.

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Higher Special Functions

A Theory of the
Central Two-Point Connection Problem
Based on a Singularity Approach

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Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

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477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment, a department of the University of Cambridge.

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www.cambridge.org
Information on this title: www.cambridge.org/9781009123198

DOI: 10.1017/9781009128414

When citing this work, please include a reference to the DOI 10.1017/9781009128414

First published 2024

A catalogue record for this publication is available from the British Library

A Cataloging-in-Publication data record for this book is available from the Library of Congress

ISBN 978-1-009-12319-8 Hardback

To
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Ann-Sophie
Clara Ruth Emilia

Cambridge University Press & Assessment
978-1-009-12319-8 — Higher Special Functions
Wolfgang Lay
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Preface

This book presents a mathematical method for calculating previously unknown functions. These functions have specific properties; therefore, they are called **special functions**. The most outstanding of their properties is that they are particular solutions of certain differential equations – linear, ordinary, homogeneous differential equations of second order with polynomial coefficients:

$$P_0(z)\,\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + P_1(z)\,\frac{\mathrm{d}y}{\mathrm{d}z} + P_2(z)\,y = 0, \quad z \in \mathbb{C}; \tag{1}$$

here $P_i(z)$, $i = 0, 1, 2$ are polynomials in z.

As is well known, such differential equations determine – as a consequence of their order – a two-dimensional set of functions. That is, among those functions which satisfy the equation, only after two quantities have been determined is one of them uniquely determined. Usually, these two quantities are the value $y(z_0)$ and the derivative $\left.\frac{\mathrm{d}y}{\mathrm{d}z}\right|_{z=z_0}$ of the function $y(z)$ at a certain point $z = z_0$ of the differential equation (1).

Now, for the differential equation (1) there are inevitably points z_1 in the domain of definition $z \in \mathbb{C}$ where the function $P_0(z_1)$ has the value zero. Such points are called **singularities of the differential equation (1)**, in contrast to the **ordinary points of the differential equation** at which $P_0(z)$ is not zero. The special thing about the singularities $z = z_1$ of the differential equation (1) is that – under specific conditions that will be discussed – already the determination of the function value $y(z_1) = y_1$ is sufficient to uniquely determine the solution among all possible functions determined by the differential equation (1). One does not have the possibility to choose both the function value and the derivative of its solution $y(z)$ at the singularity $z = z_1$ of the differential equation (1).

However, it can also happen that solutions of the differential equation (1) become singular at their singularities $z = z_1$, i.e., that the function value increases over all limits when approaching the singularity. However, it is a characteristic of linear differential equations that their solutions can only become singular at their singularities: at ordinary points of the differential equation (1), all of its solutions are holomorphic.

In this singular case, a specific behaviour of the function value when approaching the singularity has to be taken, rather than the function value itself.

As is well known, two functions $y_1(z)$ and $y_2(z)$ are called **linearly independent** if there is no constant $C \in \mathbb{C}$ such that

$$y_2(z) = C\, y_1(z) \tag{2}$$

hold; otherwise, these two functions are called **linearly dependent**. It is also known that the general solution $y^{(g)}(z)$ of the differential equation (1) is given by

$$y^{(g)}(z) = C_1\, y_1(z) + C_2\, y_2(z), \tag{3}$$

where the two particular solutions $y_1(z)$ and $y_2(z)$ of the differential equation (1) are supposed to be linearly independent and the coefficients $C_1 \in \mathbb{C}$ and $C_2 \in \mathbb{C}$, which are independent of z, take all values of their domain of definition $\mathbb{C}$. Such a pair of solutions is called a **fundamental system**.

What is so special about the special functions? Let us consider two singularities of the differential equation (1), which for the sake of simplicity should lie on the positive real axis, i.e., at $z = z_1 = 0$ and at $z = z_2 = 1$. The interval in between we call the **relevant interval**. Let us now consider a solution $y(z)$ of the differential equation (1), which is supposed to have a certain function value $y(z_1) = y_1$ at the singularity at $z = z_1$. Thus, the function, and therefore the solution $y(z)$, is uniquely determined. We can no longer demand that this function should assume a certain value at z_2. This is only possible again if at least one of its coefficients $P_i(z)$, $i = 0, 1, 2$, also depends on a parameter E in addition to the independent variable z. Only then may there be certain values $E = E_i$, $i = 0, 1, 2, \ldots$, of this parameter E for which there are partial solutions $y_{E_i}(z)$ of the differential equation (1), which take a given value at both points z_1 and z_2. The computation of this parameter $E = E_i$, $i = 0, 1, 2, \ldots$, is called the boundary eigenvalue problem, and the values $E = E_i$, $i = 0, 1, 2, \ldots$, are called eigenvalues. Because at least one of the boundary points of the relevant interval is a singularity of the differential equation (1), it is not only a boundary eigenvalue problem but a **singular boundary eigenvalue problem**. If we are dealing with singular solutions of the differential equation (1), then – as already mentioned above – the function value is replaced by the behaviour when approaching the singularity radially. This is especially true if the singularity z_2 is an improper point of the equation (1), i.e., if it is placed at infinity, which can be symbolised by $z_2 = \infty$.

It becomes obvious that the key to the calculation of special functions is a method to calculate their eigenvalues. This book explains under what conditions classical methods for the determination of eigenvalues exist, and where this is not yet the case. Then, a mathematical principle is formulated and a mathematical method is developed with the help of which one can calculate the eigenvalues and, thus, the special functions for all differential equations (1). It is immediately obvious that this considerably expands the hitherto rather limited number of special functions.

Now, one could suggest that the singularities of the differential equation (1) have only a finite number of singular points of the differential equation and are, therefore, negligible, while all other points (and thus the vast majority) are ordinary points of the differential equation. However, this is not true: singularities, although they occur only once in the differential equation (1), or at least only sporadically, essentially determine the solutions in the entire definition area of the equation: the singularities of the differential equation (1) are the cornerstones of its solution. So it is not surprising that the following applies: every differential equation (1) has at least one singularity.

Special functions have enjoyed great popularity among mathematicians as well as physicists. The following anecdote has been passed down, showing the tremendous popularity of probably the most-cited book in mathematics ever. **Sir Michael V. Berry**, the English theoretical physicist, was once invited to contemplate being marooned on the proverbial desert island. He was asked what book he would most like to have there, in addition to the Bible and (as an English person) the complete works of Shakespeare. His immediate answer was: Abramowitz and Stegun's *Handbook of Mathematical Functions* (which first appeared in 1964), perhaps the most successful work of mathematical reference ever published. This answer is by no means surprising: as soon as principles dominate in a field of science, special functions become important. Hereby, special functions are standardised functions that first originate from mathematically formulated rules and second are able to describe a variety of scientific phenomena.

This book is not designed as a reference book, nor to raise any claim of completeness, but it first displays the exact solution of the central two-point connection problem and second founds the hope that in the near future the topic of higher special functions will experience a revival in science and in lecturing, yielding lots of new insights, surprising scientific results and not-yet-seen phenomena.

Chapter 1 uses the example of classical special functions to show the aspect from which they must be considered, so that mathematical methods can be derived which are so general that they can also be used to calculate the higher special functions. In addition, the underlying mathematical principle is presented.

In Chapter 2, a concept for the treatment of singularities of linear differential equations and their local solutions is presented, which goes back to Henri Poincaré, but deviates from it in one fundamental respect.

In Chapter 3, the differential equations of the Fuchs class are presented as fundamental equations from which many others – so-called confluent cases – can be derived. Based on the methodology of Chapter 2, the Fuchs equation with four singularities is presented as an example. Finally, the totality of the differential equation is presented in a schematic.

In Chapter 4, the singular boundary eigenvalue problem is formulated and the methods developed in previous chapters are calculated and solved using the differential equations presented in Chapter 3.

In Chapter 5, the previously developed method is applied concretely to selected examples. On the one hand, this shows the usefulness of the basic method as developed in previous chapters; on the other hand, it shows how much more knowledge of the concrete problem is needed in order to move on from the general level of theory to the solution of such concrete problems.

Acknowledgements

A book like the present one cannot be produced without the direct and indirect participation of a whole series of people – even and especially if it is a monograph. Such is the case here. I would, therefore, like to take this opportunity to mention a group of people who, in one way or another, have made a significant contribution to the success of this project.

I would like to thank **Professor Dr Sir Michael Victor Berry** of the University of Bristol for a research stay at the H. H. Wills Physics Laboratory as a fellow of the Max Planck Society, which brought me into contact with the topic of special functions and just at a time when the first thoughts on the DLMF (Digital Library of Mathematical Functions) project were taking shape.

I would like to thank **Professor Dr Felix Medland Arscott**(†) of the University of Winnipeg, who accompanied and guided me on my way to becoming a scientist, with a sensitivity and respect that I have rarely experienced. He was a great teacher of science and I will always honour his memory. That I have remained true to the field of differential equations is, to a large extent, due to him.

I owe special thanks to **Professor Dr Serguei Yuriewitsch Slavyanov**(†) of St. Petersburg State University. As mentor he advised and shaped me over a 30-year period of scientific cooperation. Without his scientific supervision, the creation of this book would not have been possible.

I would like to thank **Dipl. Phys. Karlheinz Bay** for scientific cooperation, and carrying out the necessary numerical calculations over several years of research on the subject.

I would like to thank **Dr Achim Dannecker** for his support over the period of writing this book, and for valuable discussions on various topics concerning the drafting of a standard work in mathematics. His mental support and active interest in the development of the work over many years have encouraged and supported me, especially during difficult phases.

Special thanks go to **Dipl. Math. Hardy Wagner**, who accompanied me on the difficult and long journey from typescript to book. With his patience and empathy, he is a real friend and an excellent mathematician.

I would like to thank **Dr Tino Lukaschek** for a critical review of the typescript and for numerous valuable suggestions in connection with the writing of the book.

I would like to thank **Sandra and Stefan Mercamp** for their mental guidance over the years, and for their interest and participation in the creation of this book. Discussions with them in a relaxed atmosphere have given me courage when the process of writing was at a standstill.

I would like to thank **Professor Dr Charles William Clark** of the National Institute of Standards and Technology (NIST) in Gaithersburg (Maryland) for his recommendation to have the book published by Cambridge University Press and to have it reviewed by Professors Peter John Olver and Mark Jay Ablowitz. The large-scale DLMF project has contributed significantly to the writing of this book.

I would like to thank **Professor Dr Peter John Olver** of the University of Minnesota (Minneapolis) and **Professor Dr Mark Jay Ablowitz** of the University of Colorado (Boulder) for their scholarly reading, for numerous valuable suggestions and – above all – for recommending the work to Cambridge University Press for publication.

I would also like to thank my wife **Dr Cornelia Charlotte Matz** and my daughters **Ann-Sophie Lay** and **Clara Ruth Emilia Lay** for their critical comments and constant consideration when it came to supporting me in my research efforts. I thank my wife for her support in compiling the reference list, and Ann-Sophie for various discussions in connection with the writing and publication of the work, and for always listening to me when I needed her advice and opinion.