978-1-009-12319-8 — Higher Special Functions
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Excerpt
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Introduction

Classical special functions are a traditional field of mathematics. As particular solutions of singular boundary eigenvalue problems of linear ordinary differential equations of second order, they are by definition functions that can be represented as the product of an asymptotic factor and a (finite or infinite) Taylor series. The coefficients of these series are by definition solutions of two-term recurrence relations, from which an algebraic boundary eigenvalue criterion can be formulated. This method is called the *Sommerfeld polynomial method* (Rubinowicz, 1972); thus, one can say that the boundary eigenvalue condition is by definition algebraic in nature. It is the central message in this book that one can resolve this restriction and it is shown how to do this methodically, and what the fundamental mathematical principle underlying this method is.

Now, of course, the method developed for this also applies to problems that can be solved with classical methods. So, in order to present the newly developed methods in the light of what is known, and to be able to understand the new perspective more easily (and also measure the results obtained against what is already known), this new method is applied in this chapter to the already known solutions. This makes it possible to classify the new according to the well known. Accordingly, it is a 'phenomenological' introduction, where the focus is not on definitions, theorems and their proofs, but on the ad hoc introduction of the relevant quantities. The systematic introduction based on definitions follows in the subsequent chapters, this time with respect to differential equations, which are no longer accessible by classical methods.

1.1 Historical Remarks

The history of the subject treated here goes back to the middle of the nineteenth century, to a time when the German mathematician **Lazarus Fuchs** (1833–1902) wrote down the local solution of a linear, ordinary differential equation with polynomial coefficients in the vicinity of a singularity. This solution consists of a product of an (in general) irrational power of the independent variable and an (in general) infinite Taylor, thus one-sided, series. Yet, the explicit form of the solution was only one aspect of the great importance of Fuchs' approach. It was just as important to have recognised the significance of the singularity of a differential equation in the first place. Singularities are something ubiquitous; they cannot be avoided, not in the equation and certainly not in the solution. And not only that: singularities of differential equations determine the behaviour of their solutions everywhere, even where the equations are holomorphic. This is perhaps the most significant peculiarity of differential equations. Therefore, it is more than justified to bring singularities to the centre of consideration if one wants to deal with differential equations in depth. This insight, which is now about 150 years old, is the basis of this work.

It is well known that the basic approach to solving a differential equation is based on Weierstrass' approximation theorem for integer powers, according to which powers can be used to approximate any holomorphic function. Based on this, the French mathematician **Paul Painlevé** (1863–1933) developed a basic existence theorem for local solutions, the so-called 'calcul des limites' or majorant method. In turn, all fundamental questions – such as those concerning analytical continuations or the uniqueness of solutions – are based on this. One can say that this complex of fundamental questions was largely settled at the beginning of the twentieth century.

The full significance of Fuchs' work was revealed in the fact that there are two types of singularities in differential equations: those that are now called *regular* and those that are called *irregular*. This distinction is triggered by the nature of their local solutions. For the irregular singularities of differential equations, the Fuchsian approach did not provide local solutions, only for regular ones. As the German mathematician **Meyer Hamburger** (1838–1903) showed a little later, this requires generalised, two-sided infinite power series, i.e., Laurent series. This makes for a significant technical complication. So, it remained to search for a one-sided series approach for the case of irregular singularities. Although the Fuchsian approach could not do this, it showed the way: a one-sided replacement for the Laurent series was then concretely worked out about 20 years later by the French mathematician **Achille Marie Gaston Floquét** (1847–1920) and, at about the same time, by the German mathematician **Ludwig Wilhelm Thomé** (1841–1910). (That is why these approaches are now called Thomé solutions in Germany and Floquét solutions in France.) The crucial idea was to write down the asymptotic factors, explicitly. The problem with Thomé's solutions, however, was that they did not converge in all cases. Thomé put a lot of effort into showing under what conditions his series approaches converged. It was finally the French mathematician **Henri Poincaré** (1854–1912) who showed the significance of Thomé's series: they are asymptotic solutions of the differential equation that represent their solutions within sectors, at the top of which is placed the singularity. The lateral lines delimiting these sectors are called *Stokes lines*, after the Irish mathematician and physicist **Sir George Gabriel Stokes** (1819–1903).

Another important insight into linear differential equations was that it was understood that differential equations with exclusively regular singularities could serve as the cornerstone of a mathematical theory, because the irregular singularities could be generated by merging regular singularities. Equations with exclusively regular singularities were given the name *Fuchsian differential equations* and the simplest forms among them were in turn given their own names. Thus, the simplest form of the Fuchsian differential equation with only one singularity is called *Laplace's equation*, the one with two singularities is called *Euler's equation* and the one with three singularities is called *Gauss' equation*.

The Fuchsian approach always leads to difference equations for its coefficients and thus to asymptotic questions of their solution for large values of the index. It has been found that significant differences in the asymptotic behaviour of the solutions can exist, although the ratio of two successive terms of the solutions tends towards one and the same value for large values of the index. Depending on whether this is the case with a difference equation or not, it is called *regular* or *irregular*.

As far as the investigation of regular difference equations is concerned, Henri Poincaré and the German mathematician **Oskar Perron** (1880–1975) excelled. For the study of irregular difference equations, in turn, the Canadian mathematician **George David Birkhoff** (1884–1944), the Russian mathematician **Waldemar Juliet Trjitzinsky** (1901–1973) and the American mathematician **Clarence Raymond Adams** (1898–1965) made lasting contributions in the 1920s (see, e.g., Birkhoff–Trjitzinsky theory and the Birkhoff–Adams theorem in Aulbach et al., 2004). Thus, while Henri Poincaré recognised the importance of Thomé's approaches to linear differential equations by proving their asymptotic character, Birkhoff and Trjitzinsky carried out the analogous investigations for linear difference equations.

The importance of all this work for the development of modern physics at the beginning of the twentieth century should be undisputed, namely relativity and quantum theory, just as this development in turn had an effect on mathematics. The realisation that nature is essentially linear on small scales, as evidenced by the *Schrödinger equation*, has given rise to this retroactive influence. Singular boundary eigenvalue problems of linear differential equations of second order were henceforth an important object of research. It turned out that it is not the order of the differential equation that determines the order of the difference equation resulting from a Fuchsian approach, but the number of its singularities.

In order for the solution of a linear differential equation to behave in a prescribed manner not only at one point, but at two, this solution must have a parameter. In quantum theory, this is the energy parameter. In order for the solution to behave in a prescribed manner at two different points, this energy parameter must assume certain values. Calculating these means solving the boundary eigenvalue problem. The condition on which this determination is based is called the boundary eigenvalue condition.

The mathematical form of the boundary eigenvalue condition is of central importance for the solution of the problem. It turns out that this condition is generally only algebraic in nature if the underlying differential equation is either of the Laplacian, Eulerian or Gaussian type, i.e., originates from a Fuchsian equation that has at most three singularities. If this number of singularities is larger than three, then the boundary eigenvalue condition becomes transcendental. And for such equations, there was only one method to write the boundary eigenvalue condition at all, and this is only if the order of the difference equation which the coefficients of Fuchs' solution approaches is at most two: the method of infinite continued fractions.

Transcendental boundary eigenvalue conditions thus occur systematically for Fuchsian differential equations that have more than three singularities. The simplest of these equations is called *Heun's differential equation*, because the German mathematician **Karl Heun** (1859–1929) studied it systematically for the first time in a paper from 1889.

Today, a distinction is made between algebraic and transcendental boundary eigenvalue conditions in two respects: the singular boundary eigenvalue problems that lead to transcendental eigenvalue conditions are called *central two-point connection problems* (CTCPs) and the resulting solution functions are called *higher special functions*; all others are called *classical special functions*.

To be precise, local solution approaches such as those of Lazarus Fuchs are no longer sufficient for solving singular boundary eigenvalue problems. This required a large-scale scientific development that began when **Arnold Sommerfeld** (1868–1951) had his great era as a professor at the Ludwig Maximilian University of Munich. In 1919, **Wolfgang Ernst Pauli** (1900–1958) asked Sommerfeld for a topic for a doctoral thesis. Pauli received from Sommerfeld the task of applying the rules of quantum mechanics established by **Niels Bohr** (1885–1962) in 1913 in order to show that the ionised hydrogen molecule is stable under the conditions of the then new quantum mechanics. Since the quantisation rules of Bohr and Sommerfeld (for which Bohr received the Nobel Prize) were not quite correct (they were what are now called semiclassical quantisation rules), he did not quite succeed, but obtained the result that the ionised hydrogen molecule was at least metastable. Thus the problem of calculating quantum mechanical energy levels became the most important in quantum physics, as it was supposed to explain the chemical bonding between two protons by a single electron.

In 1933 **George Cecil Jaffé** (1888–1965) took up the problem anew, then having available Schrödinger's differential equation and thus the correct quantisation rules. So, Jaffé was eventually able to show the stability of the ionised hydrogen molecule. However, what became much more important was that Jaffé applied a marvellous transformation of the underlying differential equation that enabled him to apply a solution ansatz that solved the problem exactly. This idea is the basis of the method of solution of the specific singular boundary eigenvalue problem, called the *central two-point connection problem* in this book. It consists of a series ansatz and a transformation that I call, in honour and commemoration of George Jaffé, the *Jaffé ansatz* and the *Jaffé transformation*, respectively. The particular solution of the underlying differential equation represented by the Jaffé ansatz is called the *Jaffé solution* (see Jaffé, 1933).

Unfortunately, George Jaffé, as a Jewish professor at the Universität Gießen, was dismissed in 1933 and in 1934 had to abandon his position as a professor, eventually going into exile in 1939, leaving Germany for ever. He went to Bâton Rouge in the US state of Louisiana but did not pick up his ideas there any more.

Sixty years after this great publication, in spring 1993, I visited the elderly **Friedrich Hund** (1896–1997) – one of the founders of quantum theory – in Göttingen. He told me the sad story of George Jaffé's fate. So this book was also written as a protest against oblivion, against the forgetting of a great scientist and his difficult life.

In 1989, Karl Heun's publication about the differential equation that nowadays bears his name celebrated its centennial appearance. In order to mark this date, an international conference was organised that brought together experts from all over the world who worked in the field. The result of this conference was several commitments to promote the topic, and even several notations that have been agreed. It was common opinion that the step from classical special functions to higher special functions was ripe to be made, and the participants were in agreement that this was a fundamental step.

Subsequently, following this conference, an unpublished conference booklet was written (Seeger and Lay, 1990), then a book written by several participants of the conference (Ronveaux, 1995), then a monograph on the topic by the Russian expert **Serguei Yuriewitsch Slavyanov** (1942–2019) and the present author (Slavyanov and Lay, 2000), and eventually a successor to the famous *Handbook of Mathematical Functions* by **Milton Abramowitz** (1915–1958) and **Irene Stegun** (1919–2008) that was edited by **Frank William John Olver** (1924–2013), entitled *The NIST Handbook of Mathematical Functions*, published (within the DLMF) by the National Institute of Standards and Technologies (NIST; Olver et al., 2010). In this book, the Heun differential equation has been dealt with as a new differential equation, not incorporated within the set of differential equations, the solutions of which belong to the classical special functions of mathematical physics (cf. Chapter 31). The authors of this part on Heun's differential equation were **Brian D. Sleeman** (1939–2021), a British professor and participant of the famous conference introduced above, and the Russian mathematician **Vadim B. Kuznetsov** (*1963).

However, the Heun equation has not been treated in an exhaustive manner, since several important topics had not been tackled or settled up to that time. These circumstances were the impetus for me to undertake a large-scale investigation in order to treat the main remaining problem: the central two-point connection problem for differential equations beyond Gaussian type. The solution of this problem in a satisfactory manner, i.e., theoretically as well as calculatory, yields the possibility of raising the whole field on a new level: singular boundary eigenvalue problems of differential equations beyond Gaussian type, stemming from applications that are solvable. Now, these mathematical problems lead to new functions (higher special functions) and show up new phenomena not yet seen, based on the fact that the differential equations have a sort of parameter that differential equations yielding classical special functions do not have.

1.2 Classical Special Functions: Testing the New in a Well-Known Area

In this section, those differential equations are described whose singular boundary eigenvalue problems lead to the *classical special functions*. It may seem superfluous to rewrite what has already been sufficiently described in books such as Whittaker and Watson (1927), Coddington and Levinson (1955), Ince (1956), Bieberbach (1965), Olver (1974) or Hille (1997). However, this is an inaccurate impression. Here, in order to get beyond the results of classical theory, one must adopt a somewhat different viewpoint than the one which developed the classical theory. This slightly different viewpoint, which allows a generalisation, must of course also be applicable to the classical theory. Thus, the known terrain is useful because it allows the new theory to be proven on the basis of known results. Anyone who picks up pencil and paper and starts calculating will appreciate being able to direct the new along the lines of the old.

1.2.1 Differential Equations

The Gauss Equation

Basically, there is one differential equation whose singular eigenvalue problems produce classical special functions: that is, the *Gauss differential equation*. It is a differential equation (1), where $P_0(z)$ is the fourth-order polynomial

$$P_0(z) = z^2\,(z-1)^2,$$

$P_1(z)$ is the third-order polynomial

$$P_1(z) = (A_0 + A_1)\,z^3 - (2\,A_0 + A_1)\,z^2 + A_0\,z$$

and $P_2(z)$ is the second-order polynomial

$$P_2(z) = (B_0 + B_1 - C)\,z^2 - (2\,B_0 - C)\,z + B_0.$$

By means of

$$P(z) = \frac{P_1(z)}{P_0(z)}, \quad Q(z) = \frac{P_2(z)}{P_0(z)}, \tag{1.2.1}$$

equation (1) on page ix becomes

$$\frac{d^2y}{dz^2} + P(z)\,\frac{dy}{dz} + Q(z)\,y(z) = 0, \quad z \in \mathbb{C} \tag{1.2.2}$$

with

$$P(z) = \frac{P_1(z)}{P_0(z)} = \frac{A_0}{z} + \frac{A_1}{z-1},$$
$$Q(z) = \frac{P_2(z)}{P_0(z)} = \frac{B_0}{z^2} + \frac{B_1}{(z-1)^2} + \frac{C}{z} - \frac{C}{z-1}. \tag{1.2.3}$$

As is seen, the coefficient $P(z)$ has a first-order pole at the singularities of the differential equation, located at $z = 0$ and at $z = 1$, and the coefficient $Q(z)$ has a second-order pole there. This is the crucial significance of the singularity of the differential equation at hand. Astonishingly, this differential equation has not just two but three singularities, two of which are placed at finite points, being called *finite singularities* for short, namely at $z = 0$ and at $z = 1$; the third one is located at infinity, being called *infinite singularity* for short. Since infinity is neither a proper point nor a number, this improper point of the differential equation (1.2.2) is dealt with by means of inverting it, viz. applying the transformation

$$\zeta = \frac{1}{z}, \tag{1.2.4}$$

thus getting

$$\frac{d^2y}{d\zeta^2} + \left[\frac{2}{\zeta} - \frac{1}{\zeta^2}\,P(\zeta)\right]\frac{dy}{d\zeta} + \frac{1}{\zeta^4}\,Q(\zeta)\;y = 0, \quad \zeta \in \mathbb{C}, \tag{1.2.5}$$

with $P(\zeta) = P(1/z)$ and $Q(\zeta) = Q(1/z)$, and then considering the point $\zeta = 0$. With equation (1.2.3) this means

$$\tilde{P}(\zeta) = \frac{2}{\zeta} - \frac{1}{\zeta^2}\,P(\zeta) = \frac{\tilde{A}_0}{\zeta} + \frac{A_1}{\zeta-1},$$
$$\tilde{Q}(\zeta) = \frac{1}{\zeta^4}\,Q(\zeta) = \frac{\tilde{B}_0}{\zeta^2} + \frac{B_1}{(\zeta-1)^2} + \frac{\tilde{C}}{\zeta} - \frac{\tilde{C}}{\zeta-1}. \tag{1.2.6}$$

As may be seen, this differential equation is just the same as (1.2.2), (1.2.3), except for the coefficients A_0, B_0, C. These do have other values, namely

$$\tilde{A}_0 = 2 - A_0 - A_1,$$
$$\tilde{B}_0 = B_0 + B_1 - C, \tag{1.2.7}$$
$$\tilde{C} = 2\,B_1 - C,$$

while A_1 and B_1 are not affected. In particular, the order of the pole at $\zeta = 0$ is the same as at $z = 0$ in (1.2.3).

All the properties of singularities of equation (1.2.2), (1.2.3) at infinity are transferred to the singularity at zero of equation (1.2.5), (1.2.6) by means of the transformation (1.2.4).

Turning to the *local solutions* at the finite singularities of the differential equation (1.2.2), (1.2.3) it was the great discovery of Lazarus Fuchs that an ansatz for the local solutions at the singularity $z = 0$ has the form (Fuchs, 1866)

$$y_0(z) = \sum_{n=0}^{\infty} a_n\, z^{n+\alpha_0} = z^{\alpha_0} \sum_{n=0}^{\infty} a_n\, z^n. \tag{1.2.8}$$

The term z^{α_0} is denoted the *asymptotic factor*. Inserting the ansatz into the differential equation and equating each power in z to zero yields

$$\begin{aligned}
&[\alpha_0\,(\alpha_0 - 1) + A_0\,\alpha_0 + B_0\,]\; a_0 = 0,\\
&[(A_0 + \alpha_0)\,(\alpha_0 + 1) + B_0]\; \alpha_1\\
&\qquad - [2\,\alpha_0\,(\alpha_0 - 1) + 2\,(A_0 + B_0) + A_1 - C]\; a_0 = 0,\\
&[1]\; a_{n+1} + [0]\; a_n + [-1]\; a_{n-1} = 0, \qquad n = 1, 2, 3, \ldots,
\end{aligned}$$

with

$$\begin{aligned}
[1] &= (n + 1 + \alpha_0)\;[(n + \alpha_0) + A_0] + B_0,\\
[0] &= -2\,(n + \alpha_0)\;[(n - 1 + \alpha_0) - (2\,A_0 + A_1)] + (C - 2\,B_0),\\
[-1] &= (n - 1 + \alpha_0)\;[(n - 2 + \alpha_0) + A_0 + A_1] + B_0 + B_1 - C.
\end{aligned}$$

This may also be written in the form

$$[\alpha_0\,(\alpha_0 - 1) + A_0\,\alpha_0 + B_0\,]\; a_0 = 0, \tag{1.2.9}$$

$$\begin{aligned}
&[(A_0 + \alpha_0)\,(\alpha_0 + 1) + B_0]\; a_1\\
&\qquad - [2\,\alpha_0\,(\alpha_0 - 1) + 2\,(A_0 + B_0) + A_1 - C]\; a_0 = 0,
\end{aligned} \tag{1.2.10}$$

$$\begin{aligned}
&\left(1 + \frac{\alpha_1}{n} + \frac{\beta_1}{n^2}\right) a_{n+1} - \left(2 + \frac{\bar{\alpha}_0}{n} + \frac{\beta_0}{n^2}\right) a_n\\
&\qquad + \left(1 + \frac{\alpha_{-1}}{n} + \frac{\beta_{-1}}{n^2}\right) a_{n-1} = 0, \quad n = 1, 2, 3, \ldots,
\end{aligned} \tag{1.2.11}$$

with

$$\begin{aligned}
\alpha_1 &= A_0 + 2\,\alpha_0 + 1, \quad \beta_1 = (A_0 + \alpha_0)\,(\alpha_0 + 1) + B_0,\\
\bar{\alpha}_0 &= 2\,A_0 + A_1 + 2\,(2\,\alpha_0 - 1),\\
\beta_0 &= (2\,A_0 + A_1)\,\alpha_0 + 2\,\alpha_0\,(\alpha_0 - 1) + 2\,B_0 - C,\\
\alpha_{-1} &= A_0 + A_1 + 2\,\alpha_0 - 3,\\
\beta_{-1} &= (A_0 + A_1)\,(\alpha_0 - 1) + \alpha_0\,(\alpha_0 - 3) + B_0 + B_1 - C + 2.
\end{aligned}$$

With $a_0 \neq 0$ ($a_0 = 0$ would mean getting the trivial solution $y(z) \equiv 0$), two values of α_0 result from (1.2.9), namely

$$\begin{aligned}
\alpha_{01} &= \frac{1 - A_0 + \sqrt{(1 - A_0)^2 - 4\,B_0}}{2},\\
\alpha_{02} &= \frac{1 - A_0 - \sqrt{(1 - A_0)^2 - 4\,B_0}}{2}.
\end{aligned} \tag{1.2.12}$$

Thus, fixing the value of the first coefficient a_0 of the series in (1.2.8) [that actually may be used for normalising the particular solution $y(z)$] allows us to calculate the value of the second coefficient a_1 from the initial condition (1.2.10). This, in turn, allows us to recursively calculate as many coefficients a_n from (1.2.11) as are needed.

As can be seen, there appear two particular solutions of the differential equation (1.2.2) from the ansatz (1.2.8). However, these are only two linearly independent solutions, i.e., a fundamental system of the differential equation, if the difference between the two characteristic exponents (1.2.12) is not an integer. Otherwise, there may appear a logarithmic term in one of the two particular solutions (see, e.g., Bieberbach, 1965, pp. 128, 136). These particular solutions are called *Frobenius solutions*. The quantity α_0 is called the *characteristic exponent*. $y_{01}(z)$ and $y_{02}(z)$ are linearly independent, thus, the pair $\{y_{01}(z),\ y_{02}(z)\}$ may also serve as a fundamental system of solutions of the differential equation (1.2.2), (1.2.3), i.e., the general solution $y^{(g)}(z)$ of (1.2.2), (1.2.3) may be written in the form

$$y^{(g)}(z) = C_1\, y_{01}(z) + C_2\, y_{02}(z),$$

where C_1 and C_2 are the totality of complex-valued constants in z. The series in (1.2.8) is convergent with radius of convergence r ranging to the neighbouring singularity of the differential equation (1.2.2), (1.2.3), thus $r = 1$. This is a consequence of the fact that for linear differential equations, all the particular solutions at ordinary points of the differential equations are holomorphic there (cf. Bieberbach, 1965, p. 5).

There are also two local solutions at the singularity $z = 1$, given by

$$y_{1i}(z) = \sum_{n=0}^{\infty} a_n\,(z-1)^{n+\alpha_{1i}} = (z-1)^{\alpha_{1i}} \sum_{n=0}^{\infty} a_n\,(z-1)^n, \quad i = 1,2,$$

with

$$\alpha_{11} = \frac{1 - A_1 + \sqrt{(1-A_1)^2 - 4\,B_1}}{2},$$
$$\alpha_{12} = \frac{1 - A_1 - \sqrt{(1-A_1)^2 - 4\,B_1}}{2}.$$

$y_{11}(z)$ and $y_{12}(z)$ are linearly independent, thus the pair $\{y_{11}(z),\ y_{12}(z)\}$ may also serve as a fundamental system of solutions of the differential equation (1.2.2), (1.2.3), i.e., the general solution $y^{(g)}(z)$ of (1.2.2), (1.2.3) may be written in the form

$$y^{(g)}(z) = C_1\, y_{11}(z) + C_2\, y_{12}(z),$$

where C_1 and C_2 are not specific constants but are the totality of complex-valued constants in z.

Last but not least, the characteristic exponents of the Frobenius solutions at infinity are given by

$$\alpha_{\infty 1} = \frac{\tilde{A}_0 + \tilde{A}_1 - 1 + \sqrt{[\tilde{A}_0 + \tilde{A}_1 - 1]^2 - 4\,(\tilde{B}_0 + \tilde{B}_1 + \tilde{C}_1)}}{2},$$
$$\alpha_{\infty 2} = \frac{\tilde{A}_0 + \tilde{A}_1 - 1 - \sqrt{[\tilde{A}_0 + \tilde{A}_1 - 1]^2 - 4\,(\tilde{B}_0 + \tilde{B}_1 + \tilde{C}_1)}}{2},$$

with $\tilde{A}_0$ and $\tilde{B}_0$ from (1.2.7) and

$$\tilde{A}_1 = A_1, \tilde{B}_1 = B_1, \tilde{C}_1 = C.$$

Eventually it should be mentioned that the sum of all the characteristic exponents is given by

$$\alpha_{01} + \alpha_{02} + \alpha_{12} + \alpha_{12} + \alpha_{\infty 2} + \alpha_{\infty 2} = 1.$$

It may be seen from (1.2.12) that, if B_0 vanishes, thus $B_0 = 0$, one of the two characteristic exponents vanishes as well, namely

$$\alpha_{02} = 0.$$

This, in turn, means that one of the two Frobenius solutions is holomorphic at the singularity $z = 0$ of the differential equation (1.2.2), (1.2.3), and thus may be represented by a pure Taylor series. The analogous happens with respect to the singularity at $z = 1$ in the case when $B_1 = 0$. This is an important mathematical mechanism for dealing with the singular boundary eigenvalue problem in §1.2.4.

The Single Confluent Case of the Gauss Equation

The differential equation

$$\frac{d^2y}{dz^2} + \left[\frac{A_0}{z} + G_0\right]\frac{dy}{dz} + \left[\frac{B_0}{z^2} + \frac{C}{z} + D_0\right] y(z) = 0, \quad z \in \mathbb{C}, \tag{1.2.13}$$

is called the *single confluent case of the Gauss equation* since it may be derived from the Gauss differential equation (1.2.2), (1.2.3)

$$\frac{d^2y}{dz^2} + \left[\frac{A_0}{z} + \frac{A_1}{z-1}\right]\frac{dy}{dz} + \left[\frac{B_0}{z} + \frac{B_1}{(z-1)^2} + \frac{C}{z} - \frac{C}{z-1}\right] y(z) = 0, \quad z \in \mathbb{C},$$

in such a way that the singularity at $z = 1$ is considered to be located at an arbitrary point $z = z_0$ that is driven to infinity:

$$\frac{d^2y}{dz^2} + \left[\frac{A_0}{z} + \frac{A_1}{z-z_0}\right]\frac{dy}{dz} + \left[\frac{B_0}{z} + \frac{B_1}{(z-z_0)^2} + \frac{C}{z} - \frac{C}{z-z_0}\right] y(z) = 0, \quad z \in \mathbb{C},$$

with $z_0 \to \infty$, thus by carrying out a limiting process. However, the result of this limiting process depends on how it is carried out, for it has to be recognised that in order to get the most general result, the coefficients A_0, A_1, B_0, B_1, C have to be dependent on z_0. Taking the constants A_0, A_1, B_0, B_1, C as independent of z_0 does not lead to the most general result. To correctly carry out this limiting process, one has