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# HARMONIC FUNCTIONS AND RANDOM WALKS ON GROUPS

Research in recent years has highlighted the deep connections between the algebraic, geometric, and analytic structures of a discrete group. New methods and ideas have resulted in an exciting field, with many opportunities for new researchers. This book is an introduction to the area from a modern vantage point. It incorporates the main basics, such as Kesten's amenability criterion, the Coulhon and Saloff-Coste inequality, random walk entropy and bounded harmonic functions, the Choquet–Deny theorem, the Milnor–Wolf theorem, and a complete proof of Gromov's theorem on polynomial growth groups.

The book is especially appropriate for young researchers, and those new to the field, accessible even to graduate students. An abundance of examples, exercises, and solutions encourage self-reflection and the internalization of the concepts introduced. The author also points to open problems and possibilities for further research.

**Ariel Yadin** is Professor in the Department of Mathematics at Ben-Gurion University of the Negev, Israel. His research is focused on the interplay between random walks and the geometry of groups. He has taught a variety of courses on the subject and has been part of a new wave of investigation into the structure of spaces of unbounded harmonic functions on groups.

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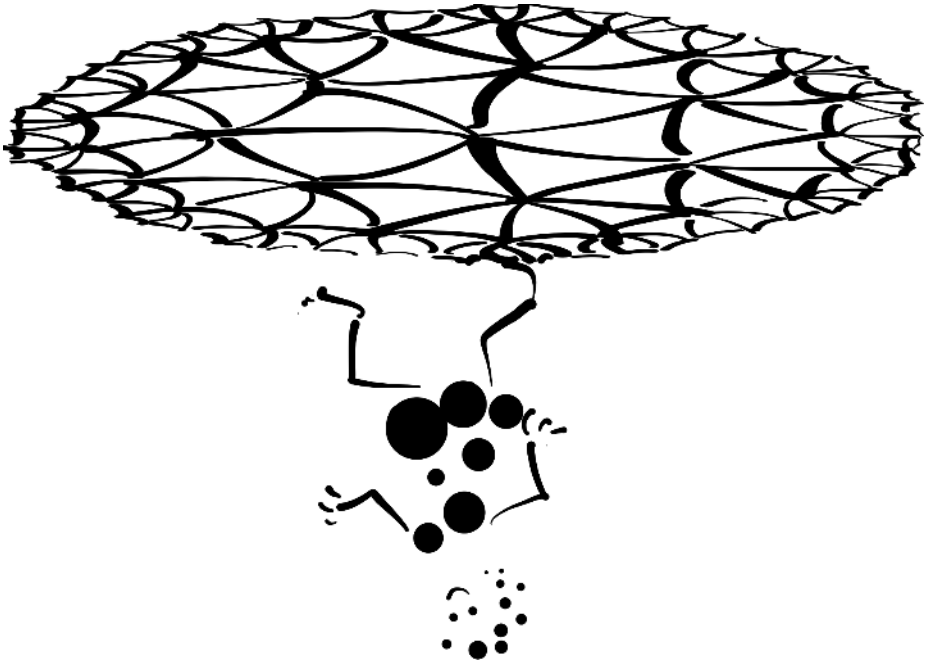

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ARIEL YADIN
*Ben-Gurion University of the Negev*

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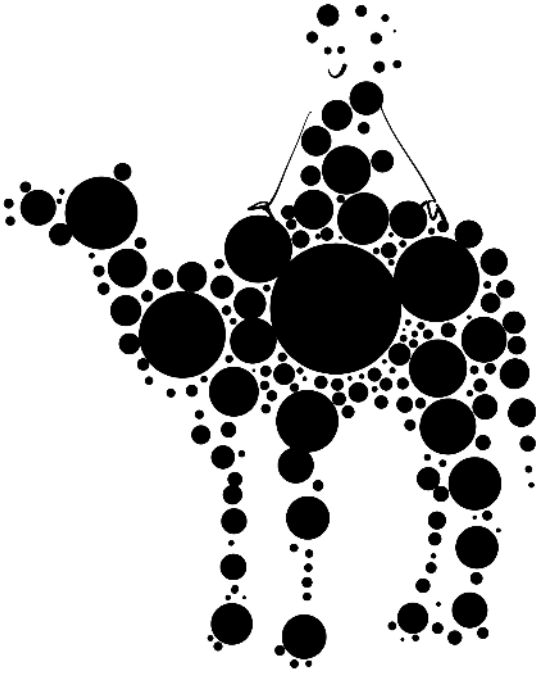



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