## Contents

**List of Figures**  
xxix

**Foreword**  
xxvii

**Preface**  
xxix

1 **Partial Differential Equations**  

1.1 Introduction  
1

1.2 Diffusion Equation  
4

1.3 First-order Equations  
6

1.4 First-order Equations: Method of Characteristics  
9

1.5 Second-order Quasilinear PDEs: Classification Using Method of Characteristics  
10

1.6 Wave Equation  
15

1.7 Linear Advection Equation  
17

1.8 Laplace Equation  
18

1.9 Method of Separation of Variables for the One-dimensional Heat Equation  
19

1.10 Method of Separation of Variables for the One-dimensional Wave Equation  
20

**Exercises**  
21

2 **Equations of Fluid Motion**  

2.1 Introduction  
25

2.2 Lagrangian and Eulerian Description of Fluid Motion  
26

2.2.1 Substantive or total derivative  
26

2.2.2 Conservation of mass principle: Continuity equation  
28

2.2.3 Conservation of momentum principle: Momentum equation  
29

2.2.4 Euler’s equation of motion for an ideal fluid  
31

2.2.5 Conservation of energy principle: Thermodynamic energy equation  
32
## Table of Contents

### 2.3 Equations Governing Atmospheric Motion
- 2.3.1 Rotating frame of reference 34
- 2.3.2 Conservation of energy: Thermodynamic energy equation for atmosphere 36
- 2.3.3 Geostrophic balance equations 38
- 2.3.4 Hydrostatic balance equation 38
- 2.3.5 Governing equations of motion of atmosphere with pressure as a vertical coordinate 39
- 2.3.6 Quasi-geostrophic equations of motion of atmosphere with pressure as a vertical coordinate 40
- 2.3.7 Shallow water equations 41
- 2.3.8 Vorticity equation for incompressible fluid: Curl of the Navier–Stokes equation 42
- 2.3.9 Vorticity equation for atmospheric and oceanic flows 43
- 2.3.10 Non-divergent vorticity equation for atmospheric and oceanic flows 43
- 2.3.11 Boussinesq approximation 43
- 2.3.12 Anelestic approximation 45
- 2.3.13 Conservation of water vapour mixing ratio equation 45
- 2.3.14 Mean equations of turbulent flow in the atmosphere 46
- 2.3.15 RANS, LES, and DNS approaches 48
- 2.3.16 Parameterization of physical processes in the atmospheric models 51
- 2.3.17 Parallel computing 51

### Exercises

### 3 Finite Difference Method
- 3.1 Introduction 57
- 3.2 Method of Finite Difference
  - 3.2.1 Forward difference scheme 59
  - 3.2.2 Backward difference scheme 60
  - 3.2.3 Central difference scheme 60
  - 3.2.4 Centered fourth-order difference scheme 61
  - 3.2.5 Finite difference scheme for second derivatives and Laplacian 61
- 3.3 Time Integration Schemes
  - 3.3.1 Two-time level schemes 65
  - 3.3.2 Three-time level schemes 67
# Contents

- Exercises 68
- Python examples 70

## 4 Consistency and Stability Analysis 73

### 4.1 Consistency and Stability Analysis 73

#### 4.2 Basic Aspects of Finite Differences 73

- 4.2.1 Consistency 74
- 4.2.2 Convergence 74
- 4.2.3 Lax Equivalence Theorem 75

### 4.3 Errors and Stability Analysis 75

- 4.3.1 Introduction 75
- 4.3.2 Discretization error 75
- 4.3.3 Representation of real numbers in a computer: Round-off error 76
- 4.3.4 Stability analysis of FTCS scheme as applied to one-dimensional heat conduction equation 78
- 4.3.5 Richardson central in time and central in space (CTCS) finite difference scheme and its stability 79
- 4.3.6 DuFort–Frankel finite difference scheme and its stability 80
- 4.3.7 Backward in time and central in space (BTCS) scheme and its stability 81
- 4.3.8 Crank–Nicolson Scheme and its stability 82

### 4.4 Two-dimensional Heat Conduction Equation 84

- 4.4.1 FTCS scheme and its stability 84
- 4.4.2 BTCS scheme and its stability 85
- 4.4.3 Alternating Direction Implicit (ADI) method 86

### 4.5 Stability Analysis of One-dimensional Linear Advection Equation 86

- 4.5.1 Forward in time and central in space (FTCS) scheme 87
- 4.5.2 Central in time and central in space (CTCS) scheme and its stability 87
- 4.5.3 Upwind methods 90
- 4.5.4 Lax finite difference scheme and its stability 91
- 4.5.5 Lax–Wendroff scheme and its stability 92
- 4.5.6 Backward in time and central in space (BTCS) scheme and its stability 93
- 4.5.7 Crank–Nicolson scheme and its stability 94

### 4.6 Matrix Method of Stability Analysis 94

- 4.6.1 Matrix method for the one-dimensional heat equation 95
### Contents

4.7 Energy Method of Stability Analysis 97  
4.8 Aliasing and Nonlinear Computational Instability 99  
4.9 Aliasing Error and Instability 100  
4.10 Ways to Prevent Nonlinear Computational Instability 102  
4.11 Arakawa’s Scheme to Prevent Nonlinear Computational Instability 103  

*Exercises*  

*Python examples* 116

#### 5 Oscillation and Decay Equations

5.1 Introduction 131  
5.2 Properties of Time-differencing Schemes as Applied to the Oscillation Equation 131  
5.3 Properties of Various Two-time Level Differencing Schemes 133  
5.3.1 Forward Euler scheme 133  
5.3.2 Backward Euler scheme 134  
5.3.3 Trapezoidal scheme 134  
5.3.4 Iterative two-time level scheme 134  
5.3.5 Matsuno scheme 135  
5.3.6 Heun scheme 135  
5.3.7 Phase change of the various two-level schemes 136  
5.4 Properties of Various Three-time Level Differencing Schemes 137  
5.4.1 Leapfrog scheme 137  
5.4.2 Adams–Bashforth scheme 139  
5.5 Properties of Various Schemes as Applied to the Friction Equation 139  
5.5.1 Application of various two-time level schemes to the friction equation 140  

*Exercises* 142

#### 6 Linear Advection Equation

6.1 Introduction 146  
6.2 Centered Time and Space Differencing Schemes for Linear Advection Equation 146  
6.3 Conservative Finite Difference Methods 150  
6.3.1 Leapfrog scheme 154
# Table of Contents

6.3.2 Matsuno scheme 158  
6.3.3 Lax–Wendroff scheme 158  
6.4 Computational Dispersion: Phase Speed Dependence on Wavelength 161  
6.4.1 Group velocity 163  
6.5 Upstream Schemes 166  
6.5.1 Transportive property 167  
6.6 Fourth-order Space Differencing Schemes for Advection Equation 170  
6.7 Higher Order Sign Preserving Advection Schemes 172  
6.8 Two-dimensional Linear Advection Equation 175  
6.8.1 Computational dispersion: Phase speed dependence on frequency 176  

**Exercises**  
177  
**Python examples**  
181  

7 Numerical Solution of Elliptic Partial Differential Equations 184  
7.1 Introduction 184  
7.1.1 Commonly occurring elliptic problems 185  
7.2 Direct Methods of Solution 186  
7.3 Iterative Methods of Solution 186  
7.3.1 Gauss–Seidel method 187  
7.3.2 Successive over relaxation (SOR) method 187  
7.3.3 Relaxation, sequential relaxation, and successive relaxation methods 188  
7.4 Multigrid Methods 197  
7.4.1 Understanding the two-grid method 198  
7.4.2 Full multigrid (FMG) method 201  
7.5 Fast Fourier Transform Methods 202  
7.6 Cyclic Reduction and Factorization Methods 205  

**Exercises**  
210  
**Python examples**  
213  

8 Shallow Water Equations 217  
8.1 Introduction 217  
8.2 One-dimensional Linear Gravity Wave without Rotation 220
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.3 Staggered Grid Arrangement for Linear One-dimensional Gravity Wave</td>
<td>226</td>
</tr>
<tr>
<td>8.4 Linear Inertia–gravity Waves in One-dimension</td>
<td>227</td>
</tr>
<tr>
<td>8.4.1 Non-staggered grid arrangement – Grid ‘A’</td>
<td>229</td>
</tr>
<tr>
<td>8.4.2 Staggered grid arrangements – Grid ‘B’</td>
<td>230</td>
</tr>
<tr>
<td>8.4.3 Staggered grid arrangements – Grid ‘C’</td>
<td>231</td>
</tr>
<tr>
<td>8.4.4 Staggered grid arrangements - Grid ‘D’</td>
<td>232</td>
</tr>
<tr>
<td>8.5 Two-dimensional Linear Gravity Wave without Rotation</td>
<td>234</td>
</tr>
<tr>
<td>8.5.1 Non-staggered grid arrangement (Grid ‘A’)</td>
<td>236</td>
</tr>
<tr>
<td>8.5.2 Staggered grid arrangement (Grid ‘B’)</td>
<td>238</td>
</tr>
<tr>
<td>8.5.3 Staggered grid arrangement (Grid ‘C’)</td>
<td>239</td>
</tr>
<tr>
<td>8.5.4 Staggered grid arrangement (Grid ‘D’)</td>
<td>241</td>
</tr>
<tr>
<td>8.5.5 Staggered grid arrangement (Grid ‘E’)</td>
<td>242</td>
</tr>
<tr>
<td>8.6 Two-dimensional Linear Gravity Wave with Rotation</td>
<td>245</td>
</tr>
<tr>
<td>8.6.1 Non-staggered grid arrangement (Grid ‘A’)</td>
<td>246</td>
</tr>
<tr>
<td>8.6.2 Staggered grid arrangement (Grid ‘B’)</td>
<td>248</td>
</tr>
<tr>
<td>8.6.3 Staggered grid arrangement (Grid ‘C’)</td>
<td>250</td>
</tr>
<tr>
<td>8.6.4 Staggered grid arrangement (Grid ‘D’)</td>
<td>253</td>
</tr>
<tr>
<td>8.6.5 Staggered grid arrangement (Grid ‘E’)</td>
<td>254</td>
</tr>
<tr>
<td>Exercises</td>
<td>259</td>
</tr>
<tr>
<td>9 Numerical Methods for Solving Shallow Water Equations</td>
<td>264</td>
</tr>
<tr>
<td>9.1 Introduction</td>
<td>264</td>
</tr>
<tr>
<td>9.2 Linear One-dimensional Shallow Water Equations without Rotation</td>
<td>264</td>
</tr>
<tr>
<td>9.3 Solution of Linear One-dimensional Shallow Water Equations</td>
<td>265</td>
</tr>
<tr>
<td>without Rotation</td>
<td>265</td>
</tr>
<tr>
<td>9.3.1 Explicit schemes: Leapfrog scheme (non-staggered)</td>
<td>265</td>
</tr>
<tr>
<td>9.3.2 Explicit schemes: FTCS scheme (non-staggered)</td>
<td>267</td>
</tr>
<tr>
<td>9.3.3 Fully implicit schemes (non-staggered)</td>
<td>267</td>
</tr>
<tr>
<td>9.3.4 Forward–backward scheme (non-staggered)</td>
<td>268</td>
</tr>
<tr>
<td>9.3.5 Pressure averaging scheme (non-staggered)</td>
<td>269</td>
</tr>
<tr>
<td>9.3.6 Implicit scheme (non-staggered)</td>
<td>270</td>
</tr>
<tr>
<td>9.3.7 Staggered explicit scheme</td>
<td>270</td>
</tr>
<tr>
<td>9.3.8 Splitting method</td>
<td>272</td>
</tr>
<tr>
<td>9.3.9 Semi-implicit method</td>
<td>274</td>
</tr>
<tr>
<td>9.3.10 Stability of the semi-implicit method</td>
<td>275</td>
</tr>
<tr>
<td>9.4 Two-dimensional Linear Shallow Water Equations without Rotation</td>
<td>277</td>
</tr>
</tbody>
</table>
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.4.1 Leapfrog scheme</td>
<td>277</td>
</tr>
<tr>
<td>9.4.2 Eliassen grid</td>
<td>278</td>
</tr>
<tr>
<td>9.4.3 Forward backward scheme</td>
<td>278</td>
</tr>
<tr>
<td>9.4.4 Implicit scheme (trapezoidal method)</td>
<td>280</td>
</tr>
<tr>
<td>9.5 Semi-implicit Scheme of Kwizak and Robert</td>
<td>281</td>
</tr>
</tbody>
</table>

### Exercises

- Exercises 283
- Python examples 287

## 10 Numerical Methods for Solving Barotropic Equations

- Introduction 313
- Numerical Solution of a Non-divergent Barotropic Vorticity Equation on a $\beta$ Plane – Linear Case 314
- Numerical Solution of a Non-divergent Barotropic Vorticity Equation on a $f$ Plane – Nonlinear Case 316
- Numerical Solution of a Non-divergent Barotropic Vorticity Equation on a $\beta$ Plane – Nonlinear Case 320
- Solving One-dimensional Linear Shallow Water Equations without Rotation 322
- Solving One-dimensional Linear Shallow Water Equations with Rotation 322
- Solving One-dimensional Nonlinear Shallow Water Equations without Rotation 323
- Solving Two-dimensional Linear Shallow Water Equations without Rotation 325
- Solving Two-dimensional Linear Shallow Water Equations with Rotation on a $\beta$ Plane 326
- Solving Two-dimensional Nonlinear Shallow Water Equations without Rotation 327
- Solving Two-dimensional Nonlinear Shallow Water Equations with Rotation on a $\beta$ Plane 328
- Equivalent Barotropic Model 329

### Exercises

- Exercises 332
- Python examples 334
# Contents

## 11 Numerical Methods for Solving Baroclinic Equations

11.1 Introduction

11.2 Atmospheric Vertical Coordinates

11.3 Pressure as a Vertical Coordinate

11.4 Sigma (\(\sigma\)) as a Vertical Coordinate

11.5 Eta (\(\eta\)) as a Vertical Coordinate

11.6 Isentropic Vertical Coordinate

11.7 Vertical Staggering

11.8 Two-layer Quasi-geostrophic Equation

11.9 Multi-level Models

11.10 Limited Area Primitive Equation Atmospheric Model

11.10.1 Finite difference equations for the limited area primitive equation atmospheric model

11.10.2 Solution procedure

**Exercises**

## 12 Boundary Conditions

12.1 Introduction

12.2 Upper Boundary Conditions

12.3 Lower Boundary Conditions

12.4 Lateral Boundary Conditions

12.5 One-way and Two-way Interactive Nesting

**Exercises**

## 13 Lagrangian and Semi-Lagrangian Schemes

13.1 Introduction

13.2 Fully Lagrangian Scheme

13.3 Semi-Lagrangian Scheme

13.3.1 Linear one-dimensional advection equation with constant velocity

13.3.2 Semi-Lagrangian scheme to solve the linear one-dimensional advection equation with constant velocity

13.3.3 Semi-Lagrangian scheme to solve the linear one-dimensional advection equation with non-constant velocity – three time level scheme
## Contents

13.3.4 Semi-Lagrangian scheme to solve the linear one-dimensional advection equation with non-constant velocity – two time level scheme 412
13.3.5 Semi-Lagrangian scheme to solve the linear one-dimensional advection equation with non-constant velocity in the presence of a source term using two time level scheme 416
13.3.6 Stability of the semi-Lagrangian scheme to solve the linear one-dimensional advection equation 417
13.3.7 Semi-Lagrangian scheme to solve the forced advection equation with non-constant velocity – three time level scheme 419
13.4 Numerical Domain of Dependence 421
13.5 Semi-Lagrangian Scheme to Solve Shallow Water Equations 422
  13.5.1 Advantages of semi-Lagrangian scheme as compared to Eulerian scheme 424
13.6 Interpolation 424
Exercises 426

14 Spectral Methods 428
  14.1 Introduction 428
14.2 Series Expansion Method 428
14.3 Spectral Methods and Finite Difference Method 430
  14.3.1 Spectral methods as applied to a linear one-dimension advection equation 431
  14.3.2 Spectral methods as applied to a linear second-order ordinary differential equation 432
  14.3.3 Spectral methods as applied to a partial differential equation involving time 433
  14.3.4 Spectral methods and energy conservation 434
  14.3.5 Spectral methods applied to nonlinear one-dimensional advection equation 435
  14.3.6 Spectral methods applied to nonlinear one-dimensional advection equation – handling the nonlinear term 436
  14.3.7 Spectral methods applied to barotropic vorticity equation on a $\beta$ plane 438
  14.3.8 Types of truncation 441
  14.3.9 Advantages of spectral methods over the method of finite differences 443
## Table of Contents

14.3.10 Transform method 445
14.4 Spectral Methods for Shallow Water Equations 446
14.5 Pseudo-spectral Methods 450

*Exercises*

*Python examples*

### 15 Finite Volume and Finite Element Methods 457

15.1 Introduction 457
15.2 Integral Form of Conservation Law 457
  - 15.2.1 Integral form of conservation law of mass 458
  - 15.2.2 Convection equation 459
  - 15.2.3 Integral form of linear momentum conservation equation 459
15.3 Finite Volume Method 460
  - 15.3.1 Finite volume method applied to one-dimensional scalar conservation equation 460
  - 15.3.2 Godunov scheme for a scalar equation 464
  - 15.3.3 Rankine–Hugoniot jump condition 466
  - 15.3.4 Finite volume method for one-dimensional linear heat equation 467
15.4 Finite Element Method 471
  - 15.4.1 Finite element method as applied to an ordinary differential equation 473
  - 15.4.2 Finite element method as applied to one-dimensional advection equation 477
  - 15.4.3 Finite element method as applied to one-dimensional linear Helmholtz equation 479

*Exercises*

### 16 Ocean Models 484

16.1 Introduction 484
16.2 Sverdrup Model for Ocean Circulation 485
  - 16.2.1 Sverdrup model for ocean circulation having a zonal wind stress with meridional variation 489
16.3 Stommel Model for Ocean Circulation 490
  - 16.3.1 Stommel model for ocean circulation having a zonal wind stress with meridional variation 493
16.4 Munk Model for Ocean Circulation 494
   16.4.1 Munk model for ocean circulation having a wind stress with both zonal and meridional variations 495
16.5 Nonlinear Model for Ocean Circulation 497
16.6 Vertical Coordinate for Ocean Models 499
   16.6.1 Height (z) coordinate 499
   16.6.2 Isopycnic coordinate 499
   16.6.3 Sigma (σ) coordinate 500
   16.6.4 Hybrid coordinate 501
16.7 Barotropic–Baroclinic Splitting 501
   16.7.1 External and internal waves 501
   16.7.2 Barotropic–baroclinic subsystems 502
16.8 Time Discretization 503
   16.8.1 Leapfrog scheme 503
   16.8.2 Two-time level finite difference scheme 503
16.9 Spatial Discretization and Horizontal Grids in Ocean Models 504
   16.9.1 Spatial arrangement of the dependent variables in the horizontal 504
16.10 Various Approaches to Solving the Ocean Momentum Equations 505

Exercises 505

Appendix: Tridiagonal Matrix Algorithm 509

Bibliography 511

Index 515