1
The Fourier and Laplace Transforms

The Laplace transform is a mathematical operation that converts a function from one domain to another. And why would you want to do that? As you'll see in this chapter, changing domains can be immensely helpful in extracting information from the mathematical functions and equations that describe the behavior of natural phenomena as well as mechanical and electrical systems. Specifically, when the Laplace transform operates on a function $f(t)$ that depends on the parameter t, the result of the operation is a function $F(s)$ that depends on the parameter s. You'll learn the meaning of those parameters as well as the details of the mathematical operation that is defined as the Laplace transform in this chapter, and you'll see why the Fourier transform can be considered to be a special case of the Laplace transform.

The first section of this chapter (Section 1.1) shows you the mathematical definition of the Laplace transform followed by explanations of phasors, spectra, and the Fourier Transform in Section 1.2. You can see how these transforms work in Section 1.3, and you can view transforms from the perspective of linear algebra and inner products in Section 1.4. The relationship between the Laplace frequency-domain function $F(s)$ and the Fourier frequency spectrum $F(\omega)$ is presented in Section 1.5, and inverse transforms are described in Section 1.6. As in every chapter, the final section (Section 1.7) contains a set of problems that you can use to check your understanding of the concepts and mathematical techniques presented in this chapter. Full, interactive solutions to every problem are freely available on the book's website.

1.1 Definition of the Laplace Transform

This section is designed to help you understand the answers to the questions "What is the Laplace transform?," and "What does it mean?" As stated above, the Laplace transform is a mathematical operation that converts a function of one domain into a function of a different domain; recall that the domain of a function is the set of all possible values of the input for that function. The domains relevant to the Laplace transform are usually called the "t" domain and the "s" domain; in most applications of the Laplace transform the variable t represents time and the variable s represents a complex type of frequency, as described below. The Laplace transform is an integral transform, which means that the process of transforming a function $f(t)$ from the t-domain into a function $F(s)$ in the s-domain involves an integral:

$$F(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{+\infty} f(t)e^{-st}\,dt. \tag{1.1}$$

So what does this equation tell you? It tells you how to find the s-domain function $F(s)$ that is the Laplace transform of the time-domain function $f(t)$. In the center portion of this equation, the expression $\mathcal{L}[f(t)]$ represents the Laplace transform as a "Laplace transform operator" ($\mathcal{L}$) that takes in the time-domain function $f(t)$, performs a series of mathematical operations on that function, and produces the s-domain function $F(s)$. Those operations are shown in the right portion of the equation to be multiplication of $f(t)$ by the complex exponential function e^{-st} and integration of the product over time. The reasons for these operations are fully explained below.

You should be aware that Eq. 1.1, in which the integration is performed over all time, from $t = -\infty$ to $t = +\infty$, is the bilateral (also called the "two-sided") version of the Laplace transform. In many practical applications, particularly those involving initial-value problems and "causal" systems (for which the output at any time depends only on inputs from earlier times), you're likely to see the Laplace transform equation written as

$$F(s) = \int_{0^-}^{+\infty} f(t)e^{-st}\,dt, \tag{1.2}$$

in which the lower limit of integration is set to zero (actually 0^-, which is the instant just before time $t = 0$, as explained below) rather than $-\infty$. This is called the unilateral or "one-sided" version of the Laplace transform, and it is the form of the Laplace transform most often used in applications such as those described in Chapter 4.

If this is the first time you've encountered a zero with a minus-sign superscript (0^-), don't worry; the meaning of 0^- and why it's used as the lower limit of the unilateral Laplace transform are not hard to understand. The value of 0^- is defined by the equation

$$0^- = \lim_{\epsilon \to 0} (0 - \epsilon),$$

in which ϵ is a vanishingly small increment. Put into words, when applied to time, 0^- represents the time just before (that is, on the negative side or "just to the left" of) time $t = 0$.

Likewise, 0^+ is defined by the equation

$$0^+ = \lim_{\epsilon \to 0} (0 + \epsilon),$$

which means that 0^+ is just after (that is, on the positive side or "just to the right" of) time $t = 0$.

And here is the relevance of this to the unilateral Laplace transform: if you integrate a function using an integral with 0^- as the lower limit and any positive number as the upper limit, the value of the function at time $t = 0$ contributes to the integration process. But if the lower limit is 0^+ and the upper limit is positive, the value of the function at $t = 0$ does not contribute to the integration. Since several applications of the unilateral Laplace transform involve functions with values that change significantly when their argument equals zero, integrating such functions using 0^- as the lower limit ensures that the value at $t = 0$ contributes to the result.[1]

The Laplace transform equation may look a bit daunting at first glance, but like many equations in physics and engineering, it becomes comprehensible when you take the time to consider the meaning of each of its terms. To do that, a good place to start is to make sure that you understand the answer to the question that confounds many students even after they've learned to use the Laplace transform to solve a variety of problems. That question is "What exactly does the parameter s in the Laplace transform represent?".

As mentioned above, in most applications of the Laplace transform in physics, applied mathematics, and engineering, the real variable t in the function $f(t)$ represents time and the complex variable s in the function $F(s)$ is a generalized frequency that encompasses both a rate of decay (or growth) as well as a frequency of oscillation. Like any complex number, the s-parameter in the Laplace transform can be written as the sum of a real part and an imaginary part:

$$s = \sigma + i\omega, \tag{1.3}$$

in which σ represents the real part and ω represents the imaginary part of the complex variable s, and i represents the imaginary unit $i = \sqrt{-1}$.

[1] A complete discussion of the importance of 0^- to the Laplace transform can be found in Kent H. Lundberg, Haynes R. Miller, and David L. Trumper "Initial Conditions, Generalized Functions, and the Laplace Transform," http://math.mit.edu/~hrm/papers/lmt.pdf.

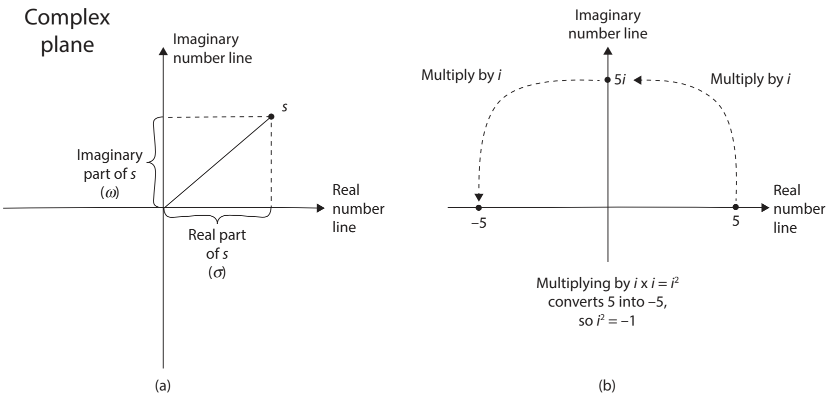

Figure 1.1 (a) Laplace parameter s on the complex plane, and (b) why $i^2 = -1$.

To understand the nature of a complex number, it's helpful to graphically represent the real part and the imaginary part of a complex number on two different number lines, as shown in the "complex plane" diagram in Fig. 1.1a. As you can see, in the complex plane the imaginary number line is drawn perpendicular to the real number line.

So what does this have to do with the imaginary unit i? Consider this: if you multiply a number on the real number line, such as 5, by the imaginary unit i, that real number 5 becomes the imaginary number $5i$ because it's now on the imaginary number line, as shown in Fig. 1.1b. And if you then multiply again by i, you get $5i \times i = -5$. So if multiplying by $i \times i$ converts the number 5 into -5, then i^2 must equal -1, which means that i must equal $\sqrt{-1}$. Since squaring any real number can't result in a negative value, it's understandable that i has come to be called the imaginary unit.

The imaginary part of s (that is, ω) is the same angular frequency that you may have encountered in physics and mathematics courses, which means that ω represents the rate of angle change, with dimensions of angle per unit time and SI units of radians per second. Since radians are dimensionless (being the ratio of an arc length to a radius), the units of radians per second (rad/sec) are equivalent to units of 1/seconds (1/sec), and this means that the result of multiplying the angular frequency ω by time (t) is dimensionless.[2] That is reassuring since $st = (\sigma + i\omega)t$ appears in the exponent of the term e^{st} in the Laplace transform.

[2] Note that the abbreviation "sec" is used for "seconds" rather than the standard "s"; this is done throughout this book in order to avoid confusion with the Laplace generalized frequency parameter.

Since ω has dimensions of 1/time and SI units of 1/sec, Eq. 1.3 makes sense only if σ and s also have dimensions of 1/time and SI units of 1/sec. For this reason, you can think of s as a "generalized frequency" or "complex frequency," in which the imaginary part is the angular frequency ω. The meaning of the real part (σ) of the Laplace parameter s, and why it's reasonable to call it a type of frequency, is explained in Section 1.5.

If you're wondering about the dimensions of the Laplace transform output $F(s)$, note that the time-domain function $f(t)$ can represent any quantity that changes over time, which could be voltage, force, field strength, pressure, or many others. But you know that $F(s)$ is the integral of $f(t)$ (multiplied by the dimensionless quantity e^{-st}) over time, so the dimensions of $F(s)$ must be those of $f(t)$ multiplied by time. Thus if $f(t)$ has dimension of voltage (SI units of volts), the Laplace transform $F(s)$ has dimensions of volts multiplied by time (SI units of volts-seconds). But if $f(t)$ has dimensions of force (SI units of newtons), then $F(s)$ has dimensions of force multiplied by time (SI units of newtons-seconds).

1.2 Phasors and Frequency Spectra

Before getting into the reasons for using a complex frequency parameter in a Laplace transform, it's helpful to consider the product of the angular frequency ω with the imaginary unit $i = \sqrt{-1}$ and time t in the exponential function $e^{i\omega t}$. This produces a type of spinning arrow sometimes called a "phasor," a contraction of the words "phased vector." As time passes, the phasor represented by $e^{i\omega t}$ rotates in the anticlockwise direction about the origin of the complex plane with angular frequency ω. As you can see in the upper left portion of Fig. 1.2, the angle (in radians) that this phasor makes with the positive real axis at any time t is given by the product ωt, so the larger the value of ω, the faster the phasor rotates. And if ω is negative, the phasor $e^{i\omega t}$ rotates in the clockwise direction.[3]

You should also remember the relationship between the exponential function $e^{i\omega t}$ and the functions $\sin(\omega t)$ and $\cos(\omega t)$ is given by Euler's relation:

$$e^{\pm i\omega t} = \cos(\omega t) \pm i \sin(\omega t), \tag{1.4}$$

[3] Some students, thinking of frequency as some number of cycles per second, wonder "How can anything rotate a *negative* number of cycles per second?" That question is answered by thinking of frequency components in terms of phasors that can rotate either clockwise (negative ω) or anticlockwise (positive ω).

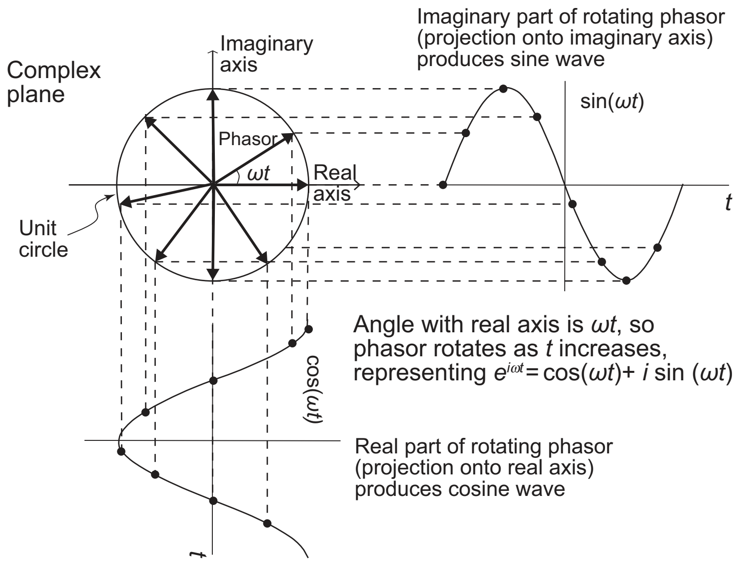

Figure 1.2 Rotating phasor relation to sine and cosine functions.

which is also illustrated in Fig. 1.2. As shown in the figure, the projection of the rotating phasor onto the real axis over time traces out a cosine function, and the phasor's projection onto the imaginary axis traces out a sine function. Adding a time axis as the third dimension of the complex-plane graph of a rotating phasor results in the three-dimensional plot shown in Fig. 1.3, in which the solid line represents the path of the tip of the $e^{i\omega t}$ phasor over time.

So the exponential function $e^{i\omega t}$ is complex, with the real part equal to $\cos(\omega t)$ and the imaginary part equal to $\sin(\omega t)$. Also helpful are the inverse Euler relations, which tell you that the cosine and sine functions can be represented by the combination of two counter-rotating phasors ($e^{i\omega t}$ and $e^{-i\omega t}$):

$$\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2} \tag{1.5}$$

and

$$\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}. \tag{1.6}$$

(If you'd like to see how these counter-rotating phasors add up to give the cosine and sine functions, check out the problems at the end of this chapter and the online solutions).

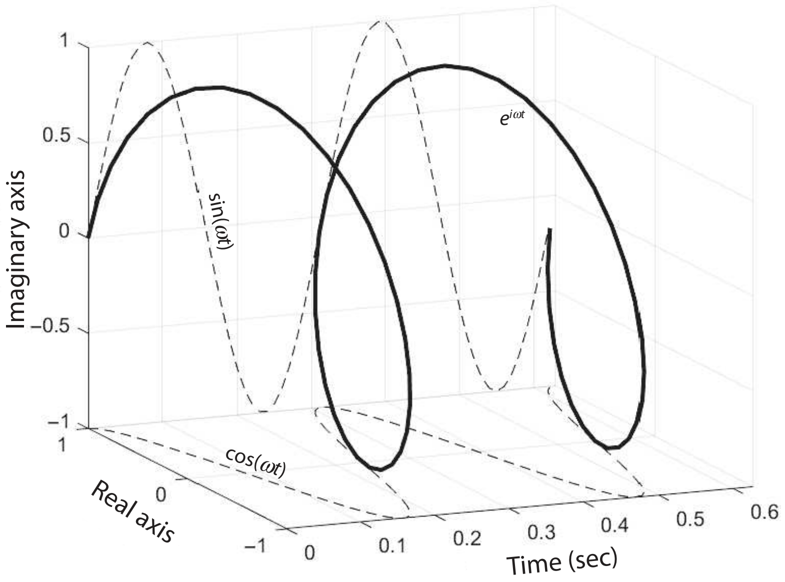

Figure 1.3 The path of the tip of the rotating phasor $e^{i\omega t}$ over time with $\omega = 20$ rad/sec.

This means that the real angular frequency ω does a perfectly good job of representing a sinusoidally varying function when inserted into the exponential function $e^{i\omega t}$. But that may cause you to wonder "Why bother making a complex frequency $s = \sigma + i\omega$?"

To understand the answer to that question, it's helpful to begin by making sure you understand the Fourier transform, which is a slightly simpler special case of the Laplace transform. The equation for the Fourier transform looks like this:

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt. \tag{1.7}$$

Comparing this equation to the equation for the bilateral Laplace transform (Eq. 1.1), you can see that these equations become identical if you set the real part σ of the Laplace complex frequency s to zero, which makes $s = \sigma + i\omega = 0 + i\omega$. That makes the bilateral Laplace transform look like this:

$$F(s) = \int_{-\infty}^{+\infty} f(t)e^{-(\sigma+i\omega)t}dt = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt, \tag{1.8}$$

which is identical to the Fourier transform. So the Fourier transform is a special case of the Laplace transform; specifically, it's the case in which the real part σ of the complex frequency s is zero.[4]

Just as the Laplace transform tells you how to find the s-domain function $F(s)$ that is the Laplace transform of the time-domain function $f(t)$, the Fourier transform tells you how to find the frequency-domain function $F(\omega)$ (often called the "frequency spectrum") that is the Fourier transform of the time function $f(t)$. Note that although $f(t)$ may be purely real, the presence of the factor $e^{-i\omega t}$ means that the frequency spectrum $F(\omega)$ may be complex.

There are many helpful books and online resources dealing with the Fourier transform, including one of the books in the *Student's Guide* series (*A Student's Guide to Fourier Transforms* by J. F. James), so you should take a look at those if you'd like more detail about the Fourier transform. But the remainder of this section contains a short description of the meaning of the frequency spectrum $F(\omega)$ produced by operating on a time-domain function $f(t)$ with the Fourier transform, and the next section has an explanation of why the operations shown in Eq. 1.7 produce the frequency spectrum $F(\omega)$, along with an example.

It is important to understand that the frequency-domain function $F(\omega)$ contains the same information as the time-domain function $f(t)$, but in many cases the frequency-domain representation may be much more readily interpreted. That is because the frequency spectrum $F(\omega)$ represents a time-changing quantity (such as a voltage, wave amplitude, or field strength) not as a series of values at different points in time, but rather as a series of sinusoidal "frequency components" that add together to produce the signal or waveform represented by the time-domain function $f(t)$. Specifically, for a real time-domain function $f(t)$, at any angular frequency ω, the real part of $F(\omega)$ tells you how much $\cos(\omega t)$ is present in the mix, and the imaginary part of $F(\omega)$ tells you how much $\sin(\omega t)$ is present. And although the function $f(t)$ that represents the changes in a quantity over time can look quite complicated when graphed, that behavior may be produced by a mixture of a reasonably small number of sinusoidal frequency components. An example of that is shown in Fig. 1.4, in which $f(t)$ represents a time-varying quantity and $F(\omega)$, the Fourier transform of $f(t)$, is the corresponding frequency spectrum (for simplicity, only the positive-frequency portion of the spectrum is shown). As you can see, trying to determine the frequency content using the graph of $f(t)$

[4] Note that this does not mean that you can find the Fourier transform $F(\omega)$ of any function simply by substituting $s = i\omega$ into the result of the Laplace transform $F(s)$ – for that to work, the region of convergence of the Laplace transform must include the $\sigma = 0$ axis in the complex plane, as discussed in Section 1.5.

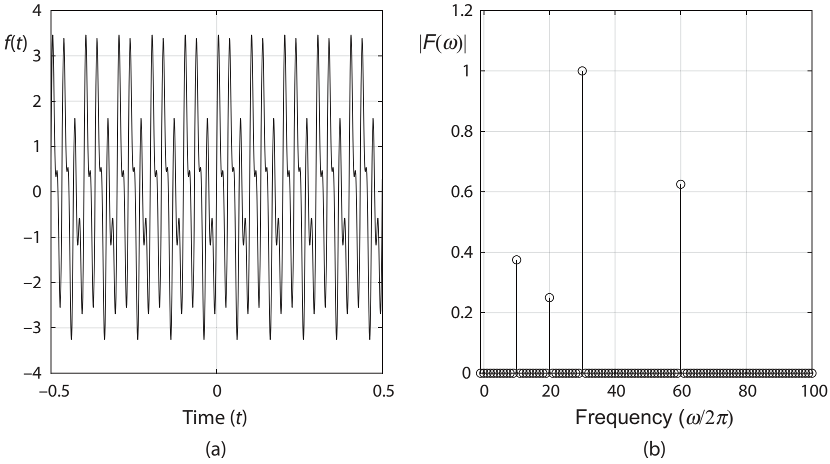

Figure 1.4 (a) Time-domain function $f(t)$ and (b) magnitude of frequency-domain function $F(\omega)$.

shown in Fig. 1.4a would be quite difficult. But the graph of the magnitude of the frequency-domain function $|F(\omega)|$ in Fig. 1.4b makes it immediately obvious that there are four frequency components present in this waveform. The frequency of each of those four components is given by its position along the horizontal axis; those frequencies are 10, 20, 30, and 60 cycles per second (Hz) in this case. The height of each peak indicates the "amount" of each frequency component in the mix that makes up $f(t)$; in this case those relative amounts are approximately 0.38 at 10 Hz, 0.23 at 20 Hz, 1.0 at 30 Hz, and 0.62 at 60 Hz.

So that's why it is often worth the effort to calculate the frequency spectrum $F(\omega)$ by taking the Fourier transform of $f(t)$. But how exactly does multiplying $f(t)$ by the complex exponential $e^{-i\omega t}$ and integrating the product over time accomplish that?

You can get a sense of how the Fourier transform works by remembering that multiplying $f(t)$ by $e^{-i\omega t}$ is just like multiplying $f(t)$ by $\cos(\omega t)$ and by $-i \sin \omega t$, so the Fourier transform can be written like this:

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt$$

$$= \int_{-\infty}^{+\infty} f(t)\cos(\omega t)dt - i\int_{-\infty}^{+\infty} f(t)\sin(\omega t)dt.$$

This form of the Fourier transform reminds you that when you use the Fourier transform to generate the frequency spectrum $F(\omega)$, you are essentially "decomposing" the time-domain function $f(t)$ into its sinusoidal frequency components (a series of cosine and sine functions that produce $f(t)$ when added together in proper proportions). To accomplish that decomposition, you can use the functions $\cos(\omega t)$ and $\sin(\omega t)$ for every value of ω as "testing functions" – that is, functions that can be used to determine whether these frequency components are present in the function $f(t)$. Even better, the process of multiplication by these testing functions and integration of the product over time is an indicator of "how much" of each frequency component is present in $f(t)$ (that is, the relative amplitude of each cosine or sine function). You can see how that process works in the next section.

1.3 How These Transforms Work

To understand how the process of decomposing a time-domain function into its sinusoidal frequency components works, consider the case in which the time-domain function $f(t)$ is simply a cosine function with angular frequency ω_1. Of course, for this single-frequency case you can see by inspection of a graph of $f(t)$ that this function contains only one frequency component, a cosine function with angular frequency ω_1, but the decomposition process works in the same way when a cosine or sine function is buried among many other components with different frequencies and amplitudes.

Figures 1.5 and 1.6 show what happens when you multiply $f(t) = \cos(\omega_1 t)$ by cosine and sine functions (the real and imaginary parts of $e^{-i\omega t}$). In Fig. 1.5,

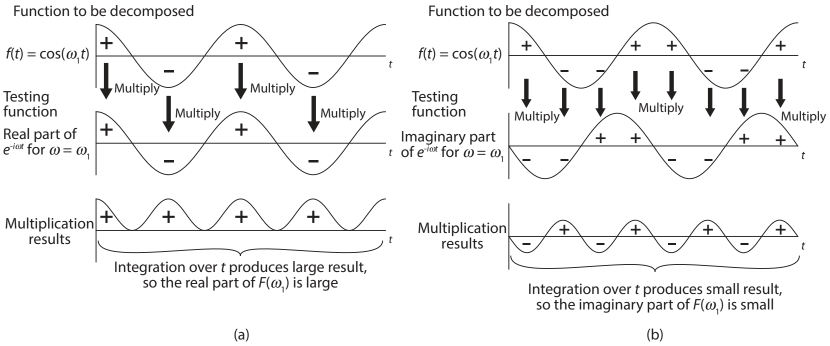

Figure 1.5 Multiplying $f(t) = \cos(\omega_1 t)$ by (a) the real portion and (b) the imaginary portion of $e^{-i\omega t}$ when $\omega = \omega_1$.