COMPOUND RENEWAL PROCESSES

Compound renewal processes (CRPs) are among the most ubiquitous models used in applications of probability. At the same time, they are a natural generalization of random walks, the most well-studied classical objects in probability theory. This monograph, written for researchers and graduate students, presents the general asymptotic theory and generalizes many well-known results concerning random walks. The book contains the key limit theorems for CRPs, functional limit theorems, integro-local limit theorems, large and moderately large deviation principles for CRPs in the state space and in the space of trajectories, including large deviation principles in boundary crossing problems for CRPs, with an explicit form of the rate functionals, and an extension of the invariance principle for CRPs to the domain of moderately large and small deviations. Applications establish the key limit laws for Markov additive processes, including limit theorems in the domains of normal and large deviations.

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Compound Renewal Processes

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Introduction

The Objects of Study

Compound renewal processes (CRPs) are among the most common mathematical models in many applications of probability theory, such as queuing theory, insurance theory, risk theory, and others. They are also used in theoretical research, for example, in the study of Markov additive processes (see [95, 96]). At the same time they are a natural generalization of random walks, the most well-studied classical objects in probability theory. The new general asymptotic theory of CRPs constructed in this monograph is therefore both of applied interest and generalizes many well-known results of probability theory related to random walks (see, e.g., A.A. Borovkov's monograph [23]). Some of the results obtained in this monograph for CRPs turned out to be new for the special case of random walks as well (see, e.g., Chapter 7).

Suppose we are given a random vector (τ_1, ζ_1) and a sequence of independent identically distributed random vectors (τ, ζ), (τ_2, ζ_2),. . . independent of (τ_1, ζ_1), where $\tau_1 \geq 0$, $\tau > 0$. We set

$$T_n := \sum_{j=1}^{n} \tau_j, \qquad Z_n := \sum_{j=1}^{n} \zeta_j \quad \text{for } n \geq 1, \qquad T_0 = Z_0 = 0. \tag{1}$$

For $t \geq 0$, we put

$$\eta(t) := \min\{k \geq 0 : T_k > t\}, \qquad \nu(t) := \eta(t) - 1. \tag{2}$$

Clearly,

$$\nu(t) = \max\{k \geq 0 : T_k \leq t\}$$

for all $t \geq 0$.

The random processes $\eta(t)$ and $\nu(t)$ are called *renewal processes* (or *simple renewal processes*).

The term "renewal process" first appeared in connection with "technical" applied problems in which failures and renewals of some devices such as electrical gadgets are present, the lifetimes of the devices usually being random (see, e.g., [53]). Let $\tau_1, \tau_2, \ldots$ be the trouble-free operation times of the devices. After time τ_j, a *failure* occurs and the faulty device is subject to *renewal* (or replacement). Assume that renewal (replacement) occurs instantly. Then $\nu(t)$ will be the *number of renewals* that occur before time t if we do not count the "renewal" at time $t = 0$. The number of renewals will be equal to $\eta(t)$ if we assume that a renewal occurs at $t = 0$.

In almost all applied problems, the properties of the distribution of τ_1 depend on when we start observing the system. If we know that a renewal occurs at time $t = 0$, then we can assume that $\tau_1, \tau_2, \dots$ are identically distributed. But if we start observing at some time and we do not know when the last renewal occurred, then it is natural to assume that the time τ_1 until the first renewal after the beginning of observation has in general a distribution different from the distribution of the intervals $\tau_2, \tau_3, \dots$ between subsequent renewals.

We will now introduce a wider class of processes.

Definition 1 The process

$$Z(t) := Z_{\nu(t)}, \qquad t \geq 0 \tag{3}$$

is called a *compound* (or *generalized*) *renewal process* (a *CRP*).

The sequence $\{(\tau_j, \zeta_j)\}$ will be called the *governing sequence of the CRP*.

As we have already mentioned, CRPs arise as a mathematical model in many applied problems, for example, in queuing theory and insurance theory. The standard generally accepted model of a CRP assumes that the time τ_1 of the first jump and the size ζ_1 of this jump have a joint distribution in general different from the joint distribution of (τ, ζ) (see, e.g., [53], [5]). This is the case, for example, for a class of CRPs important from an applications point of view, those with *stationary increments* (see §1.1.2). If $(\tau_1, \zeta_1) \underset{d}{=} (\tau, \zeta)$, then the process $Z(t)$ is called a *homogeneous CRP*; otherwise, it is called *inhomogeneous*.

The trajectories of $Z(t)$ on $[0, \infty)$ for $\tau_1 > 0$ have the following form:

$$Z(t) = \begin{cases} 0 & \text{if} \quad t \in [0, \tau_1), \\ \zeta_1 & \text{if} \quad t \in [\tau_1, T_2), \\ Z_2 & \text{if} \quad t \in [T_2, T_3) \quad \text{and so on;} \end{cases} \tag{4}$$

they are right-continuous. If $\tau_1 = 0$ (this is not excluded since $\tau_1 \geq 0$ by assumption), then the set $[0, \tau_1)$ is empty, $Z(t) = \zeta_1$ for $t \in [0, \tau_2)$, and so, the process $Z(t)$ on the event $\{\tau_1 = 0\}$ can be regarded as a homogeneous CRP with initial value $Z(0) = \zeta_1$.

Let us give two examples in which the process $Z(t)$ plays a key role in describing the work of the system under study. First, we consider the simplest problem in queuing theory. Customers with service times $\zeta_1, \zeta_2, \dots$ arrive at the queueing system at times $T_1, T_2, \dots$, respectively. For example, the customers could be airplanes arriving for landing at a busy airport: They arrive at times $T_1, T_2, \dots$, and the time needed for the landing of the jth plane is equal to ζ_j. Or the context could be an information processing system, which receives information in packets at times $T_1, T_2, \dots$, the jth packet requiring time ζ_j for its processing.

In such systems, $Z(t) = Z_{\nu(t)}$ is the time required for serving the customers received by time t. Suppose that the system under consideration is a queuing system (the customers that find the system busy join a queue). An important characteristic of such a system is the "virtual" waiting time $W(t)$ before the customer that arrives at time t starts to be serviced. It is not hard to see that $W(t)$ satisfies the equation

$$dW(t) = \begin{cases} dZ(t) - dt & \text{if} \quad W(t) > 0, \\ dZ(t) & \text{if} \quad W(t) = 0, \end{cases}$$

which has an explicit solution

$$W(t) = Z(t) - t - \inf_{0 \le u \le t} (0, Z(u) - u) \tag{5}$$

(the trajectory of $W(t)$ is obtained from the trajectory of $Z(t) - t$ by means of a "stopping" barrier at point 0). If, for example, τ_j and ζ_j are independent, then (5) implies that the limiting distribution of $W(t)$ as $t \to \infty$ coincides with the distribution of $\sup_{0 \le t < \infty} (Z^{(\mathrm{st})}(t) - t)$ (see, e.g., [17, §6]), where $Z^{(\mathrm{st})}(t)$ is a CRP with stationary increments, i.e., the process $Z(t)$ with a specially chosen distribution of τ_1 (see §1.1.2).

The second example is concerned with the operations of an insurance company. Let $T_1, T_2, \ldots$ be the times of significant claim payouts and let $\zeta_1, \zeta_2, \ldots$ be the amounts of these payments, respectively. Further, let r be the premium rate (the amount received by the company from insured customers per time unit). If x is the company's initial surplus, then its surplus at time t is equal to $x + rt - Z(t)$. This means that if $\inf_{u \le t} (x + ru - Z(u)) < 0$, then the company will go bankrupt by time t. In other words, the probability of ruin prior to time t is equal to $\mathbf{P}\big(\sup_{u \le t} (Z(u) - ru) > x\big)$. This is the classical ruin probability problem, which is the subject of many publications including monographs (see, e.g., [9], [4]–[6], [63]). It is considered in §6.7.

In both examples, the objects of study are the probabilities that the trajectory of the process $Z(u)$ crosses some boundary prior to time t. Problems of this kind are called boundary crossing problems for the CRP. They are considered in §1.6 and in Chapters 4 and 6.

CRPs also appear in theoretical research. For example, they emerge when studying the asymptotic laws for Markov additive processes (sums of random variables defined on the states of Markov chains). If the chain is Harris, then it has a positive atom, sometimes an "artificial" one. If we construct cycles (of respective lengths $\tau_1, \tau_2, \ldots$) generated by the returns of the chain to a positive atom and denote by $\zeta_1, \zeta_2, \ldots$ the increments of the sums on these cycles, then we obtain independent identically distributed vectors (τ_j, ζ_j), which define the corresponding CRP $Z(n)$ (time $t = n$ is discrete in this case). Using this process and the results presented in this monograph, one can obtain all the main limit laws for Markov additive processes (see §§1.8, 2.5, 5.7, and the references therein).

Alongside the CRP $Z(t)$, we will also consider stochastic processes

$$Y(t) := Z_{\eta(t)} = Z_{\nu(t)} + \zeta_{\eta(t)}, \qquad t \ge 0. \tag{6}$$

They will also be referred to as CRPs. The trajectories of $Y(t)$ on $[0,\infty)$ have the following form for $\tau_1 > 0$:

$$Y(t) = \begin{cases} \zeta_1 & \text{if} \quad t \in [0,\tau_1), \\ Z_2 & \text{if} \quad t \in [\tau_1, T_2) \quad \text{and so on.} \end{cases}$$

When $\tau_1 = 0$, there are changes similar to those mentioned after (4). We will show that the limit laws in the domain of normal deviations to be studied in Chapters 1 and 2, under appropriate conditions, are the same for the CRPs $Z(t)$ and $Y(t)$. In the domain of large deviations (see Chapters 3 and 6), this is not always the case.

Since $\eta(t)$ is a Markov time, the processes $Y(t) = Z_{\eta(t)}$ have a somewhat simpler structure, and, in a number of cases, it is more convenient to study these processes.

If $\tau_1 \equiv \tau \equiv 1$, then the CRP

$$Z(t) = Z([t]) = Z_{[t]}$$

becomes the *sequence of partial sums* $Z_{[t]}$ of the random variables ζ_j, i.e., a *random walk*. This object has been extensively studied.

If $\zeta_1 \equiv \zeta \equiv 1$, then $Z(t) = \nu(t) = \eta(t) - 1$, where $\eta(t)$ is a *simple renewal process*. For this process, $\mathbf{P}(\eta(t) > n) = \mathbf{P}(T_n \le t)$, and the problem of studying the distribution of $\eta(t)$ reduces to studying the distribution of sums of random variables – this time, of the sums T_n. Clearly, similar observations are also valid for the CRP $Y(t)$.

If $(\tau_1, \zeta_1) \underset{d}{=} (\tau, \zeta)$, while τ and ζ are independent,

$$\mathbf{P}(\tau > v) = e^{-\lambda v}, \qquad v \ge 0, \quad \lambda > 0, \tag{7}$$

then the process $Z(t)$ becomes a *compound Poisson process*, i.e., a process with independent increments.

If, instead of (7), we have

$$\mathbf{P}(\tau = k) = (1-q)q^k, \quad q \in (0,1), \quad k = 0, 1, \ldots,$$

or $\mathbf{P}(\tau = 1) = 1$, then, as in the case of (7), the sequence $Z(k)$ is a process with independent increments but in discrete time, i.e., a sequence of partial sums of independent identically distributed random variables (a random walk).

A Short History and the Contents of the Book

The study of CRPs is the topic of many publications. A number of general results are known such as the strong law of large numbers, the central limit theorem (see, e.g., the textbook by A.A. Borovkov [21, §10.6]), the law of the iterated logarithm, the invariance principle (see M. Csörgo, L. Hervatt, and J. Steinebach [55], [112]; A. Gut [72, Chapter 5]). The proofs in these publications rely upon a very complex technique and are simplified in Chapter 1. In the monograph [38, Chapter 16] by A.A. Borovkov and K.A.Borovkov, large deviation probabilities for CRPs and their trajectories are studied in the case when the jumps of the process have distributions regularly varying at infinity.

A significant portion of the published work related to CRPs concerns applications or rather special problem formulations. The body of work is fragmented in nature and does not deal with the main directions of interest. From 2008 to 2019, a major cycle of works by the author, K.A. Borovkov, A.A. Mogul'skii, and E.I. Prokopenko appeared (many of which are joint works) devoted to limit laws for CRPs. These works served as the basis of this book.

The goal of this monograph is a systematic exposition of the asymptotic theory of CRPs in its general form. It comprises analogs and generalizations of all the main limit laws established for random walks presented, for example, in [23]. This is the first presentation of the theory in monographic literature. It is of both theoretical and applied interest.

The theory contains:

- the basic limit theorems for CRPs in the domain of normal deviations (with the functional limit theorems), including the case of infinite variance of the jumps of the process; the law of the iterated logarithm and its analogs (Chapter 1);
- integro-local limit theorems for CRPs in the domains of normal, moderately large and large deviations (Chapters 2 and 5);
- large and moderately large deviation principles for CRPs in the state space and in the space of trajectories, including large deviation principles in boundary crossing problems for CRPs, with an explicit form of deviation function (rate function) (Chapters 3 and 4);
- limit theorems describing the sharp asymptotics in boundary crossing problems for CRPs (Chapter 6);
- extension of the invariance principle for CRPs to the domain of moderately large and small deviations (Chapter 7); the results of Chapter 7 turn out to be new for random walks as well.

We apply the theory to establish the main limit laws for Markov additive processes including functional limit theorems in the domains of normal and large deviations (§§1.8, 3.6, and 5.7).

The above suggests that a significant portion of the monograph (Chapters 3–7) is concerned with studying large deviation probabilities for CRPs. Mathematically, this is the most content-rich and difficult part of the theory we present. Note that an essential role here is played by the following circumstance. It turns out that there exists a function that encapsulates all information about the asymptotic behavior of the distribution of the CRP on increasing time intervals. We found and studied it, and call it the *fundamental* function (see §3.5). For a random walk (which is a special case of CRP), the fundamental function is equal to the logarithm of the Laplace transform of the jump distribution. In the general case, the fundamental function plays the same role as this transform, but instead of explicit equalities there will be analogous asymptotic relations as time grows.

A vast literature including several monographs is devoted to large deviations problems for random processes. Its main concern is the study of "rough" asymptotics (asymptotics of the logarithms) of the probabilities of the corresponding rare events for a wide class of processes. The main approaches and a survey of the results can be found, for example, in the monographs by J. Feng and T.G. Kurtz [64] and A. Dembo and O. Zeitouni [57] (see also the bibliography therein). In our case, in studying CRPs, we use more effective and more constructive direct approaches based on knowledge of the specific nature of CRPs, defined in (3), which makes it possible to find the fundamental function. They enable us to obtain in Chapters 3–7 much more advanced results including (a) the large deviation principle (LDP) in Chapters 3 and 4, with an explicit form of the deviation rate functional which is defined by the Legendre transform of the fundamental function; (b) in Chapters 5–7, solutions to more difficult problems on the sharp asymptotics of the large deviation probabilities.

The exposition in §§1.5–1.7 and Chapters 2, 6, and 7 is based on the author's papers [25], [30]–[36]. The exposition in Chapters 3–5 relies on the author's papers [26, 34] and on joint work with A.A. Mogul'skii [39], [44]–[48] (with E.I. Prokopenko as coauthor in [48] as well). The monograph also contains a number of previously unpublished results.

In the study of the large deviation probabilities of CRPs in Chapters 3–7, we assume that Cramér's moment condition is met (fast decay at infinity of the jump distributions). For the case where the jump distributions vary regularly at infinity (slow decay), in Chapter 8, for the sake of completeness, we present a number of results from [38, Chapter 16] without proofs.

The creation of a general asymptotic theory of CRPs under Cramér's condition became possible owing to the previous work in the following three areas:

- limit theorems on the asymptotics of the renewal measure in the domain of large deviations for multidimensional random walks (see A. A. Borovkov and A. A. Mogul'skii [39, 44]);
- large deviations of multidimensional random walks (see, e.g., A.A. Borovkov's monograph [23]); in applications to CRPs, approaches related to so-called *local* large deviation principles are very useful;
- Stone's integro-local theorems for random walks (see [114, 115]; they are particularly important as research tools).

The possession of the results and techniques in these three areas is necessary for a sufficiently complete asymptotic analysis of CRPs.

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