1

Main Limit Laws in the Normal Deviation Zone

1.1 Preliminary Results

1.1.1 Convergence of Distributions and Moments of Some Functionals of CPRs

Compound renewal processes (CRPs) $Z(t)$, $Y(t)$ were defined in the introduction (see (1)–(6)). Their definition is based on the governing sequence of vectors $\{\tau_j, \zeta_j\}$, the sums

$$T_n = \sum_{j=1}^{n} \tau_j, \qquad Z_n = \sum_{j=1}^{n} \zeta_j,$$

and the renewal processes $\eta(t) = \min\{k : T_k > t\}$ and $\nu(t) = \max\{k : T_k \le t\} = \eta(t) - 1$. In this notation,

$$Z(t) = Z_{\nu(t)}, \qquad Y(t) = Z_{\eta(t)}.$$

To describe the properties of the CRPs $Z(t)$, $Y(t)$, we need some more notation. Let $\chi(t)$ be the first overshoot over level t by the random walk $\{T_k\}_{k=1}^{\infty}$,

$$\chi(t) := T_{\eta(t)} - t, \tag{1.1.1}$$

and let

$$\gamma(t) := t - T_{\nu(t)} \tag{1.1.2}$$

be the corresponding undershoot. We also set

$$\zeta(t) := \zeta_{\eta(t)}, \qquad \tau(t) := \tau_{\eta(t)},$$

so that

$$\gamma(t) + \chi(t) = \tau(t).$$

In what follows, it is always assumed that

$$\mathbf{E}\tau =: a_\tau \quad \text{and} \quad \mathbf{E}\zeta =: a_\zeta$$

both exist, and hence the "mean drift"

$$a := \frac{a_\zeta}{a_\tau}$$

of the CRP is defined. The name "mean drift" is reasonable, because we shall show that

$$\frac{\mathbf{E}Z(t)}{t} \to a, \qquad \frac{Z(t)}{t} \underset{\text{a.s.}}{\to} a \quad \text{as } t \to \infty.$$

The same relations also hold for $Y(t)$. *In what follows, the assumption that a_τ, a_ζ exist and are finite will generally not be repeated.*

The distribution of (τ_1, ζ_1) can be arbitrary; the conditions related to this vector will be specified if necessary.

Below, we will also use the *renewal function* corresponding to the sequence $\{T_k\}$; in the homogeneous case, we denote it by

$$H(t) = \sum_{k=0}^{\infty} \mathbf{P}(T_k \le t).$$

In the inhomogeneous case $\tau_1 \underset{d}{\ne} \tau$, it is denoted by $\widetilde{H}(t)$.

We have

$$\{\eta(t) > k\} = \{T_k \le t\},$$

and hence, in the homogeneous case,

$$\mathbf{E}\eta(t) = \sum_{k=1}^{\infty} \mathbf{P}(\eta(t) \ge k) = \sum_{k=0}^{\infty} \mathbf{P}(\eta(t) > k) = \sum_{k=0}^{\infty} \mathbf{P}(T_k \le t) = H(t), \tag{1.1.3}$$

$$T_{\eta(t)} = t + \chi(t)$$

and so, by the Wald identity,

$$\begin{aligned} \mathbf{E}T_{\eta(t)} &= a_\tau \mathbf{E}\eta(t) = t + \mathbf{E}\chi(t), \\ \mathbf{E}\eta(t) &= H(t) = \frac{t + \mathbf{E}\chi(t)}{a_\tau}. \end{aligned} \tag{1.1.4}$$

Lemma 1.1.1 *Let the distribution of τ be nonlattice. Then*

(i) *The proper limit distributions*

$$\lim_{t\to\infty} \mathbf{P}(\gamma(t) \ge u, \chi(t) \ge v, \zeta(t) \ge w) = \frac{1}{a_\tau} \int_u^\infty \mathbf{P}(\tau \ge y + v,\ \zeta \ge w)\, dy, \tag{1.1.5}$$

$$\lim_{t\to\infty} \mathbf{P}(\zeta(t) \ge w) = \frac{\mathbf{E}(\tau;\ \zeta \ge w)}{a_\tau} \tag{1.1.6}$$

always exist.

(ii) *In the homogeneous case, there exist constants $c_1 \in (0, \infty)$ and $c_2 \in (1/a_\tau, \infty)$ such that*

$$\sup_t \mathbf{P}(\gamma(t) \geq u, \chi(t) \geq v, \zeta(t) \geq w)$$
$$\leq c_1 \mathbf{P}(\tau \geq u+v, \zeta \geq w) + c_2 \int_u^\infty \mathbf{P}(\tau > y+v, \zeta \geq w)\, dy. \qquad (1.1.7)$$

In the inhomogeneous case, the term $\mathbf{P}(\tau_1 \geq u+v, \zeta_1 \geq w)$ *is added to the right-hand side of* (1.1.7).

(iii) *If* $\mathbf{E}\tau_1^{k-1} < \infty$ *and* $\mathbf{E}\tau^k < \infty$*, then*

$$\mathbf{E}\gamma^k(t) = o(t), \qquad \mathbf{E}\chi^k(t) = o(t) \quad \text{as} \quad t \to \infty. \qquad (1.1.8)$$

If $\mathbf{E}\tau_1^k < \infty$ *and* $\mathbf{E}\tau^{k+1} < \infty$*, then for nonlattice* τ*,*

$$\mathbf{E}\gamma^k(t) \to \frac{\mathbf{E}\tau^{k+1}}{(k+1)a_\tau}, \quad \mathbf{E}\chi^k(t) \to \frac{\mathbf{E}\tau^{k+1}}{(k+1)a_\tau} \quad \text{as} \quad t \to \infty. \qquad (1.1.9)$$

If $\mathbf{E}e^{\lambda\tau_1} < \infty$ *and* $\mathbf{E}e^{\lambda\tau} < \infty$ *for* $\lambda > 0$*, then*

$$\mathbf{E}e^{\lambda\gamma(t)} \to \frac{\mathbf{E}e^{\lambda\tau} - 1}{a_\tau \lambda} \quad \text{as} \quad t \to \infty. \qquad (1.1.10)$$

The same relation also holds for $\chi(t)$.

If f *is a measurable function and* $\mathbf{E}|f(\zeta_1)| < \infty$, $\mathbf{E}\tau|f(\zeta)| < \infty$*, then*

$$\mathbf{E}f(\zeta(t)) \to \frac{\mathbf{E}(\tau f(\zeta))}{a_\tau} \quad \text{as} \quad t \to \infty. \qquad (1.1.11)$$

The lemma shows that the limit values of the moments of the random variables $\gamma(t)$, $\chi(t)$, $\zeta(t)$ coincide with the moments of the variables γ_∞, χ_∞, ζ_∞, whose joint distribution is given by

$$\mathbf{P}(\gamma_\infty \geq u, \chi_\infty \geq v,\ \zeta_\infty \geq w) = \frac{1}{a_\tau} \int_u^\infty \mathbf{P}(\tau \geq y+v, \zeta \geq w)\, dy \qquad (1.1.12)$$

(see (1.1.5)).

If the distribution of τ is arithmetic, then the integrals in (1.1.12) are replaced by sums – this slightly changes the values of the right-hand sides in (1.1.9) and (1.1.10).

Proof of Lemma 1.1.1 (i) For homogeneous CRPs, in view of the main renewal theorem for nonlattice τ, we have

$$\mathbf{P}(\gamma(t) \geq u, \chi(t) \geq v, \zeta(t) \geq w) = \sum_{k=1}^\infty \int_0^{t-u} \mathbf{P}(T_k \in dy)\mathbf{P}(\tau \geq t-y+v, \zeta \geq w)$$
$$= \int_0^{t-u} dH(y)\mathbf{P}(\tau \geq t-y+v, \zeta \geq w) \to \frac{1}{\mathbf{E}\tau} \int_u^\infty \mathbf{P}(\tau \geq y+v, \zeta \geq w)\, dy \qquad (1.1.13)$$

as $t \to \infty$.

If the distribution of (τ_1, ζ_1) differs from the general distribution of (τ, ζ), then, for $t > u$, we have

$$\mathbf{P}(\gamma(t) \geq u, \chi(t) \geq v, \zeta(t) \geq w) = \mathbf{P}(\tau_1 \geq t + v, \zeta_1 \geq w) + \int_0^{t-u} \mathbf{P}(\tau_1 \in ds)\mathbf{P}(\gamma(t-s) \geq u, \chi(t-s) \geq v, \zeta(t-s) \geq w), \quad (1.1.14)$$

where, for each fixed s, we have the convergence as $t \to \infty$ of the form (1.1.13) for the second factor under the integral sign in (1.1.14). This means that for the inhomogeneous CRPs the left-hand side of (1.1.14) also converges as $t \to \infty$ to the right-hand side of (1.1.13). This proves (1.1.5). Putting $u = v = 0$ in (1.1.13), (1.1.14), we get (1.1.6).

(ii) For the renewal function $H(t)$, there always exist constants $c_1 > 0$ and $c_2 > 1/a_\tau$ such that

$$H(t) \leq c_1 + c_2 t \quad \text{for all} \quad t \geq 0.$$

On the other hand, the integrand $\mathbf{P}(\tau \geq t - y + v, \zeta \geq w)$ in (1.1.13) is increasing with y. Hence the left-hand side in (1.1.13), which is 0 for $t \leq u$, is majorized by

$$c_1\mathbf{P}(\tau \geq t + v, \zeta \geq w) + c_2 \int_0^{t-u} \mathbf{P}(\tau \geq t - y + v, \zeta \geq w)\, dy \quad (1.1.15)$$

for $t \geq u$. This implies (1.1.7).

The additional term on the right of (1.1.7) in the inhomogeneous case is brought about by the appearance of the first term on the right of (1.1.14).

(iii) Assertion (1.1.8) follows from (1.1.15). If $\mathbf{E}\tau^{k+1} < \infty$, then by (1.1.12) the function

$$u^k \mathbf{P}(\gamma_\infty \geq u) = \frac{u^k}{a_\tau} \int_u^\infty \mathbf{P}(\tau \geq u)\, dy$$

is integrable. Hence by assertions (i), (ii) of the theorem, we can apply the dominated convergence theorem, which implies that

$$\mathbf{E}\gamma^k(t) \to \mathbf{E}\gamma_\infty^k = \frac{1}{a_\tau} \int_0^\infty y^k \mathbf{P}(\tau \geq y)\, dy = \frac{\mathbf{E}\tau^{k+1}}{(k+1)a_\tau}.$$

The proof of the remaining relations in (1.1.9) is similar. The last assertion follows from the equality

$$\mathbf{E}\zeta(t) = \mathbf{E}(\zeta(t);\, \zeta(t) \geq 0) + \mathbf{E}(\zeta(t);\, \zeta(t) < 0) = \int_0^\infty \mathbf{P}(\zeta(t) \geq w) dw - \int_{-\infty}^0 \mathbf{P}(\zeta(t) \leq w) dw.$$

Lemma 1.1.1 is proved. □

In studying the limit distributions of the variables $\chi(t), \zeta(t)$ as $t \to \infty$, we can also consider the "triangular array scheme," when the distribution of the "inhomogeneous"

Cambridge University Press & Assessment
978-1-009-09844-1 — Compound Renewal Processes
A. A. Borovkov , Translated by Alexey Alimov
Excerpt
More Information

vector (τ_1, ζ_1) depends on some parameter N. Such schemes appear, for example, when considering the processes

$$Z_N(t) = Z(N+t) - Z(N). \tag{1.1.16}$$

The role of the initial jumps for $Z_N(t)$ will be played by the variables $\chi(N)$ and $\zeta(N)$, whose distribution depends on N.

Corollary 1.1.2 *If in a partial triangular array scheme, the parameter $N = N(t)$ depends on t so that $\tau_1 = o_p(t)$ as $t \to \infty$, then assertions* (1.1.5) *and* (1.1.6) *of* Lemma 1.1.1 *remain valid. If, moreover,* $\mathbf{E}(\tau_1; \tau_1 \geq t) \to 0$ *and* $\mathbf{E}(\zeta_1; \tau_1 \geq t) \to 0$ *as $t \to \infty$, then assertions* (1.1.9) *and* (1.1.12) *for $k = 1$, $f(z) = z$ also remain valid.*

The first assertion of the corollary follows from (1.1.14). The second one follows from (1.1.9), (1.1.11) for $\mathbf{E}\chi(t)$ and $\mathbf{E}\zeta(t)$.

Corollary 1.1.3 *If* $\mathbf{E}\tau_1 < \infty$*, then as $t \to \infty$*

$$\widetilde{H}(t) = \frac{t + o(t)}{a_\tau}.$$

This result follows from (1.1.3), Lemma 1.1.1, and the relations

$$\begin{aligned}\widetilde{H}(t) = \mathbf{E}\eta(t) &= \mathbf{P}(\tau_1 > t) + \int_0^t \mathbf{P}(\tau_1 \in ds)[1 + \mathbf{E}\eta_0(t-s)] \\ &= \mathbf{P}(\tau_1 > t) + \int_0^t \mathbf{P}(\tau_1 \in ds)\left(1 + \frac{t - s + \mathbf{E}\chi(t-s)}{a_\tau}\right),\end{aligned}$$

where the first passage time $\eta_0(t)$ corresponds to the homogeneous sequence $\{T_k\}$.

Results similar to Lemma 1.1.1 can be found, for example, in [21, §§10.4, 10.6].

1.1.2 CRPs with Stationary Increments

Let us return to the special case (1.1.16). If the initial jumps $(\tau_{1,N}, \zeta_{1,N})$ of the process $Z_N(t)$ are labeled by the index N, then, as already pointed out,

$$(\tau_{1,N}, \zeta_{1,N}) \underset{d}{=} (\chi(N), \zeta(N)),$$

so that, by Lemma 1.1.1,

$$(\tau_{1,N}, \zeta_{1,N}) \Rightarrow (\chi_\infty, \zeta_\infty) \quad \text{as} \quad N \to \infty$$

(the sign "⇒" denotes weak convergence of distributions), where the distribution $(\chi_\infty, \zeta_\infty)$ is described in (1.1.12). Consider the CRP $Z^{(\mathrm{st})}(t)$ (the meaning of the index (st) will be explained in Definition 1.1.4) with initial jumps $(\tau^{(\mathrm{st})}, \zeta^{(\mathrm{st})})$, which have the same distribution as $(\chi_\infty, \zeta_\infty)$ (see (1.1.12)),

$$\mathbf{P}(\tau_1^{(\mathrm{st})} \geq v, \zeta_1^{(\mathrm{st})} \geq w) = \frac{1}{a_\tau} \int_v^\infty \mathbf{P}(\tau \geq y, \zeta \geq w)\, dy; \tag{1.1.17}$$

the successive jumps being (τ_2, ζ_2), (τ_3, ζ_3),... By the above interpretation, for such a process, the distribution of $(\chi^{(\mathrm{st})}(t), \zeta^{(\mathrm{st})}(t))$ (with obvious agreement in notation) will be the same for all $t \geq 0$ (the distribution of $(\chi^{(\mathrm{st})}(t'), \zeta^{(\mathrm{st})}(t'))$ can be looked upon as the limit distribution of the initial jump of the process $Z_{N'}(t) = Z_{N+t'}(t)$ with $N' = N + t' \to \infty$). This means that the distribution of the increments of $Z^{(\mathrm{st})}(u+t) - Z^{(\mathrm{st})}(u)$ will be the same for all $u \geq 0$.

Definition 1.1.4 The process $Z(t)$ with the initial jump distribution (1.1.17) is called a CRP *with stationary increments* and is denoted by $Z^{(\mathrm{st})}(t)$.

Let us find the form of the characteristic function $\varphi^{(\mathrm{st})}(\lambda, \mu)$ of the vector $(\tau_1^{(\mathrm{st})}, \zeta_1^{(\mathrm{st})})$. We set

$$\varphi(\lambda, \mu) = \mathbf{E} e^{i\lambda\tau + i\mu\zeta}.$$

Lemma 1.1.5 (i) *The following representation holds*:

$$\varphi^{(\mathrm{st})}(\lambda, \mu) = \frac{\varphi(\lambda, \mu) - \varphi(0, \mu)}{i\lambda a_\tau}. \tag{1.1.18}$$

(ii) *The process $Z(t) = Z^{(\mathrm{st})}(t)$ is a homogeneous CRP if and only if $Z(t)$ is a compound (generalized) Poisson process, that is, τ and ζ are independent, and* $\mathbf{P}(\tau > x) = e^{-x/a_\tau}$.

Proof of Lemma 1.1.5 (i) In view of (1.1.17) we have

$$\begin{aligned}\varphi^{(\mathrm{st})}(\lambda, \mu) &= \int_0^\infty e^{i\lambda v} \int_{-\infty}^\infty e^{i\mu w} \mathbf{P}(\tau_1^{(\mathrm{st})} \in dv, \zeta_1^{(\mathrm{st})} \in dw) \\ &= \frac{1}{a_\tau} \int_0^\infty e^{i\lambda v} \Big[\int_{-\infty}^\infty e^{i\mu w} \mathbf{P}(\tau \geq v, \zeta \in dw) \Big] dv. \end{aligned} \tag{1.1.19}$$

We denote by $U(v)$ the expression in square brackets on the right of (1.1.19) and let $V(v) = \frac{e^{i\lambda v}}{i\lambda}$. Hence, integrating by parts (1.1.19), we find that

$$\begin{aligned}\varphi^{(\mathrm{st})}(\lambda, \mu) &= \frac{1}{a_\tau} U(v)V(v)\Big|_0^\infty + \frac{1}{i\lambda a_\tau} \int_0^\infty e^{i\lambda v + i\mu w} \mathbf{P}(\tau \in dv, \zeta \in dw) \\ &= \frac{1}{i\lambda a_\tau} [\varphi(\lambda, \mu) - \varphi^{(\zeta)}(\mu)],\end{aligned}$$

where $\varphi^{(\zeta)}(\mu) = \mathbf{E} e^{i\mu\zeta}$.

(ii) The second assertion follows from the fact that by (1.1.18) the equality

$$\varphi^{(\mathrm{st})}(\lambda, \mu) = \varphi(\lambda, \mu)$$

is equivalent to the equality

$$\varphi(\lambda, \mu) = \frac{\varphi^{(\zeta)}(\mu)}{1 - i a_\tau \lambda}.$$

The lemma is proved. □

From Lemma 1.1.1 it follows that $\mathbf{E}\tau_1^{(\mathrm{st})} < \infty$ if $\mathbf{E}\tau^2 < \infty$ (see (1.1.9)), and $\mathbf{E}\zeta^{(\mathrm{st})} < \infty$ if $\mathbf{E}\tau^2 < \infty$, $\mathbf{E}\zeta^2 < \infty$ (see (1.1.11)). If τ and ζ are independent, then $\mathbf{E}\zeta^{(\mathrm{st})} < \infty$ if $\mathbf{E}\tau < \infty$, $\mathbf{E}|\zeta| < \infty$.

1.1.3 Strong Law of Large Numbers for a Simple Renewal Process $\eta(t)$

Lemma 1.1.6 *The almost sure convergence*

$$\frac{\eta(t)}{t} \underset{a.s.}{\to} \frac{1}{a_\tau} \quad \text{as} \quad t \to \infty \tag{1.1.20}$$

always holds.

Proof of Lemma 1.1.6 Consider the function $T_t := T_{[t]}$ of the real variable t. For this function, by the strong law of large numbers

$$\frac{T_t}{t} \underset{\text{a.s.}}{\to} a_\tau,$$

as $t \to \infty$. That is, for any $\varepsilon > 0$, there exists a (random) number $t_0 = t_0(\varepsilon) < \infty$ such that T_t lies between the rays $y = (a_\tau \pm \varepsilon)t$ for all $t > t_0$. This means that the function $\eta(y) = \min\{t : T_t > y\}$, which is the inverse of T_t, lies for all $y \geq t_0(a_\tau + \varepsilon)$ between the rays

$$t = \frac{y}{a_\tau \pm \varepsilon}.$$

But this means that (1.1.20) holds. Lemma 1.1.6 is proved. □

1.1.4 Almost Sure Convergence of Some Functionals of CRPs

Consider some measurable function $g(t, y) \geq 0$ and set, for brevity,

$$g = g(\tau, \zeta), \qquad g_n = g(\tau_n, \zeta_n), \qquad \overline{g}_n = \max_{k \leq n} g_k.$$

Let $V(x)$ be some nondecreasing regularly varying function on $[0, \infty)$, that is, a function with the representation

$$V(x) = x^\alpha l(x), \qquad x \geq 0, \quad \alpha > 1, \tag{1.1.21}$$

where $l(x)$ is a slowly varying function as $x \to \infty$. We let $V^{(-1)}(y)$ denote the inverse function of $V(x)$,

$$V^{(-1)}(y) := \inf\{x : V(x) \geq y\}.$$

The function $V^{(-1)}(y)$ is also a regularly varying function (with exponent $1/\alpha$); see, for example, Theorem 1.1.3 in [38].

Similarly, if

$$F(x) = x^{-\alpha} l(x) \tag{1.1.22}$$

is a regularly varying at infinity function with exponent $-\alpha$, then the function

$$\sigma(u) := F^{(-1)}(1/u) = \inf\{x : F(x) < 1/u\}$$

is also a regularly varying at infinity function (with exponent $1/\alpha$).

Lemma 1.1.7 (i) *If* (1.1.21) *holds and* $\mathbf{E}V(g) < \infty$, *then, as* $n \to \infty$, $t \to \infty$,

$$\frac{\overline{g}_n}{V^{(-1)}(n)} \underset{a.s.}{\to} 0, \qquad \frac{\overline{g}_{\eta(t)}}{V^{(-1)}(t)} \underset{a.s.}{\to} 0. \tag{1.1.23}$$

(ii) *If* $\mathbf{P}(g \geq x) \leq cF(x)$, $c = \text{const}$, $F(x) = x^{-\alpha}l(x)$, *then, for any* $\theta > 1/\alpha$, $n \to \infty$, $t \to \infty$,

$$\frac{\overline{g}_n}{\sigma(n)(\ln n)^\theta} \underset{a.s.}{\to} 0, \qquad \frac{\overline{g}_{\eta(t)}}{\sigma(t)(\ln t)^\theta} \underset{a.s.}{\to} 0. \tag{1.1.24}$$

(iii) *If* $\mathbf{P}(g \geq x) = o(F(x))$ *as* $x \to \infty$, *then*

$$\frac{\overline{g}_n}{\sigma(n)} \underset{p}{\to} 0, \qquad \frac{\overline{g}_{\eta(t)}}{\sigma(t)} \underset{p}{\to} 0 \tag{1.1.25}$$

as $n \to \infty$, $t \to \infty$, *respectively.*

Proof of Lemma 1.1.7 (i) We first consider the homogeneous case. Let us show that

$$\frac{g_n}{V^{(-1)}(n)} \underset{a.s.}{\to} 0 \quad as \quad n \to \infty. \tag{1.1.26}$$

For any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbf{P}(g_n \geq \varepsilon V^{(-1)}(n)) \leq \int_0^\infty \mathbf{P}(g \geq \varepsilon V^{(-1)}(t))\,dt = \int_0^\infty \mathbf{P}\Big(V\Big(\frac{g}{\varepsilon}\Big) \geq t\Big)dt = \mathbf{E}V\Big(\frac{g}{\varepsilon}\Big). \tag{1.1.27}$$

Since $V(g/\varepsilon) \sim \varepsilon^{-\alpha}V(g)$ as $g \to \infty$, we have

$$\mathbf{E}V\Big(\frac{g}{\varepsilon}\Big) \leq c + 2\varepsilon^{-\alpha}\mathbf{E}V(g) < \infty, \qquad c = \text{const}. \tag{1.1.28}$$

Now (1.1.26) follows from (1.1.27), (1.1.28) and the Borel–Cantelli lemma, because only a finite number of events $\{g_n \geq \varepsilon V^{(-1)}(n)\}$ occur with probability 1.

Let us now prove the first relation in (1.1.23). From (1.1.26) it follows that there exists a random number $n_0 = n_0(\varepsilon)$ such that

$$g_n < \varepsilon V^{(-1)}(n) \quad \text{for all} \quad n \geq n_0. \tag{1.1.29}$$

Moreover, there always exists a number $m_0 \geq n_0$ such that $\overline{g}_{n_0} < \varepsilon V^{(-1)}(m_0)$, and hence, by (1.1.29),

$$\overline{g}_{n_0+1} < \varepsilon V^{(-1)}(m_0), \ldots, \overline{g}_{m_0} < \varepsilon V^{(-1)}(m_0).$$

Another appeal to (1.1.29) shows that $\overline{g}_n < \varepsilon V^{(-1)}(n)$ for all $n \geq m_0$. But this means that

$$\frac{\overline{g}_n}{V^{(-1)}(n)} \underset{\text{a.s.}}{\to} 0 \quad \text{as} \quad n \to \infty. \tag{1.1.30}$$

We have $\eta(t) \underset{\text{a.s.}}{\to} \infty$ as $t \to \infty$, and now from (1.1.30) we get

$$\frac{g_{\eta(t)}}{V^{(-1)}(\eta(t))} \underset{\text{a.s.}}{\to} 0 \quad \text{as} \quad t \to \infty.$$

Moreover, by Lemma 1.1.6,

$$\frac{\eta(t)}{t} \underset{\text{a.s.}}{\to} \frac{1}{a_\tau}, \qquad \frac{V^{(-1)}(\eta(t))}{V^{(-1)}(t)} \underset{\text{a.s.}}{\to} a_\tau^{-1/\alpha}.$$

This proves the second relation in (1.1.23).

(ii) For any $\delta > 0$ and sufficiently large $n = n_\delta$, using, for example, Theorem 1.1.3 in [38], we have, for all $n \ge n_\delta$,

$$l(\sigma(n)(\ln n)^\theta) \le (\ln n)^{\theta\delta} l(\sigma(n)).$$

Hence, for $n \ge n_\delta$,

$$\begin{aligned} F(\sigma(n)(\ln n)^\theta) &= \sigma(n)^{-\alpha}(\ln n)^{-\theta\alpha} l(\sigma(n)(\ln(n))^\theta) \\ &\le \sigma(n)^{-\alpha}(\ln n)^{-\theta(\alpha-\delta)} l(\sigma(n)) = (\ln n)^{-\theta(\alpha-\delta)} F(\sigma(n)) = \frac{(\ln n)^{-\theta(\alpha-\delta)}}{n}. \end{aligned}$$

A similar analysis shows that, for $\alpha\theta > 1$ and $\delta < \frac{\alpha\theta-1}{\theta}$, we have $\theta(\alpha-\delta) > 1$ and

$$\sum_{n=n_\delta}^{\infty} \mathbf{P}(g_n > \sigma(n)(\ln n)^\theta) \le c \sum_{n=n_\delta}^{\infty} F(\sigma(n)(\ln n)^\theta) \le c \sum_{n=n_\delta}^{\infty} \frac{(\ln n)^{-\theta(\alpha-\delta)}}{n} < \infty.$$

Therefore,

$$\frac{g_n}{\sigma(n)(\ln n)^\theta} \underset{\text{a.s.}}{\to} 0 \quad \text{for} \quad \theta > \frac{1}{\alpha}, \quad n \to \infty. \tag{1.1.31}$$

The remaining part of the proof, in which (1.1.31) is used to derive (1.1.24), is similar to the proof of assertion (i) of the lemma.

(iii) For any $\varepsilon > 0$,

$$\begin{aligned} \mathbf{P}(\overline{g}_n > \varepsilon\sigma(n)) \le n\mathbf{P}(g > \varepsilon\sigma(n)) &= o(nF(\varepsilon\sigma(n))) \\ &= o(nF(\sigma(n))) = o(1) \quad \text{as} \quad n \to \infty. \end{aligned}$$

This proves the first relation in (1.1.25). The second relation follows from the fact that

$$\frac{\overline{g}_{\eta(t)}}{\sigma(\eta(t))} \underset{p}{\to} 0 \quad \text{and} \quad \frac{\eta(t)}{t} \underset{\text{a.s.}}{\to} \frac{1}{a_\tau} \quad \text{as} \quad t \to \infty.$$

It is easily seen that the above arguments do not change if an arbitrary fixed random vector is considered in place of (τ_1, ζ_1). Lemma 1.1.7 is proved. □

From Lemma 1.1.7 we have the following result. We set

$$\overline{\chi}(t) = \max_{u \le t} \chi(u), \qquad \overline{\zeta}(t) = \max_{u \le t} \zeta(u).$$

Corollary 1.1.8 (i) $\dfrac{\overline{\chi}(t)}{t} \underset{a.s.}{\to} 0$, $\dfrac{\overline{\zeta}(t)}{t} \underset{a.s.}{\to} 0$ *as* $t \to \infty$.

(ii) *The following relations hold*:

If $\mathbf{E}\tau^2 < \infty$, *then* $\dfrac{\overline{\chi}(t)}{\sqrt{t}} \underset{a.s.}{\to} 0$.

If $\mathbf{E}\zeta^2 < \infty$, *then* $\dfrac{\overline{\zeta}(t)}{\sqrt{t}} \underset{a.s.}{\to} 0$.

Proof We have $\mathbf{E}\tau < \infty$, $\mathbf{E}|\zeta| < \infty$, and hence the first assertion follows from the first assertion of Lemma 1.1.7 with $g(t, y) = t$, $g(t, y) = |y|$, and $V(x) = x$. The second assertion follows similarly from Lemma 1.1.7 with $V(x) = x^2$. □

1.2 First Moments of the Processes $Z(t)$ and $Y(t)$. Strong Laws of Large Numbers

1.2.1 Asymptotics for First- and Second-Order Moments of $Z(t)$ and $Y(t)$

We set

$$\xi_i = \zeta_i - a\tau_i, \qquad S_n = \sum_{i=1}^{n} \xi_i = Z_n - aT_n, \tag{1.2.1}$$

so that ξ_i for $i \geq 2$ are independent copies of the random variable

$$\xi = \zeta - a\tau, \qquad \mathbf{E}\xi = 0. \tag{1.2.2}$$

Theorem 1.2.1 I. *Let* $Z(t)$, $Y(t)$ *be homogeneous CRPs.*

(i) *The following relations hold*:

$$\mathbf{E}Y(t) = a\big(t + \mathbf{E}\chi(t)\big) = at + r_Y(t), \quad r_Y(t) = o(t), \tag{1.2.3}$$

$$\mathbf{E}Z(t) = a\big(t + \mathbf{E}\chi(t)\big) - \mathbf{E}\zeta(t) = at + r_Z(t), \; r_Z(t) = o(t), \tag{1.2.4}$$

as $t \to \infty$.

(ii) *If* $\mathbf{E}\tau^2 < \infty$, *then in* (1.2.3) *the asymptotic expansion holds, in which, in the nonlattice case,*

$$r_Y(t) = \frac{a_\zeta \mathbf{E}\tau^2}{2a_\tau^2} + o(1). \tag{1.2.5}$$

If, in addition $\mathbf{E}|\tau\zeta| < \infty$, *then in* (1.2.4)

$$r_Z(t) = \frac{a_\zeta \mathbf{E}\tau^2}{2a_\tau^2} - \frac{\mathbf{E}\tau\zeta}{a_\tau} + o(1). \tag{1.2.6}$$