POP-UP GEOMETRY

Anyone browsing at the stationery store will see an incredible array of pop-up cards available for any occasion. The workings of pop-up cards and pop-up books can be remarkably intricate. Behind such designs lies beautiful geometry involving the intersection of circles, cones, and spheres, the movements of linkages, and other constructions. The geometry can be modelled by algebraic equations, whose solutions explain the dynamics. For example, several pop-up motions rely on the intersection of three spheres, a computation made every second for GPS location. Connecting the motions of the card structures with the algebra and geometry reveals abstract mathematics performing tangible calculations. Beginning with the nephroid in the 19th-century, the mathematics of pop-up design is now at the frontiers of rigid origami and algorithmic computational complexity. All topics are accessible to those familiar with high-school mathematics; no calculus required. Explanations are supplemented by 140+ figures and 20 animations.

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The Mathematics Behind Pop-Up Cards

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Preface

Bertrand Russell said: “Mathematics rightly viewed possesses not only truth but supreme beauty.” It is, alas, not always possible to “rightly view” mathematics from the perspective of a student. Often mathematics is presented as a series of pointless (and difficult) exercises. However, seeing mathematics applied to tangible, physical objects in motion can reveal a glimpse of that “supreme beauty.” It can be uplifting to open a pop-up card and see a crease in cardstock sweep out a cone, while a particular corner tracks a circle formed by the intersection of two moving spheres. One can almost hallucinate these geometric structures guiding the intricate dynamics of the pop-up structures. This is my goal: to enable the reader to see and appreciate the mathematics underlying pop-up design.

Pop-up books and cards have been around since the eighteenth century, and recently have seen a surge in popularity through the elaborate designs of pop-up masters such as Matthew Reinhart and Robert Sabuda. This book will not help you achieve design mastery, and understanding the mathematics behind pop-ups is not even necessary to become a proficient “paper engineer.” But it is satisfying to understand the mechanisms behind pop-up constructions, which gives one an appreciation of the achievements of the designers. And, most importantly for my goals, one can learn fascinating mathematics through the study of pop-up designs.

The mathematics can be surprisingly intricate. It has led to at least one Ph.D. thesis (del Rosario Ruiz, 2015)[1] and several technical academic papers. But we will only need high-school mathematics: algebra and geometry, some limited trigonometry, and no calculus. The final two chapters explore algorithms, but we make no assumption of previous exposure to aspects of computer science.

Seeing the mathematics applied to real, physical structures can be illuminating: the equations are no longer pointless exercises, but are aimed at explaining visible dynamics. Although the mathematical prerequisites are minimal, much of the reasoning is in 3D and likely novel to readers, as 3D topics are not emphasized in standard curricula. We even prove half-a-dozen theorems, uncommon in high-school instruction outside of two-column geometry proofs.

For these reasons, more than a dozen boxed explanations are sprinkled throughout the text at points where the mathematics might be new or long

[1]This is a citation of a references listed in the bibliography (p. 123).

forgotten. In addition, a list of the symbols used is included at the end of the book (p. 121).

The text includes over 40 exercises, which are marked as *Practice*, *Understanding*, or *Challenge*. A Practice exercise might be a simple calculation, whereas an Understanding exercise generally requires a thorough grasp of the preceding material. Challenge exercises either go beyond the text or might involve a substantial investment of time. We encourage the reader to read all the exercises, give them as much thought as inclinations and circumstances allow, and then turn to Chapter 8 at the end of the book where solutions to all exercises are provided.

All pop-up card dynamics are driven by the opening of the card. Understanding the dynamics is most easily achieved either by manipulating a physical model or via an animation. All the templates in the book are available through the author's website,[2] and links are provided there to more than 25 animated GIFs (O'Rourke, 2021).

We now offer a brief summary of each chapter.

Summary. The first four chapters, with some exceptions in Chapter 2, describe constructions built from just a single piece of cardstock, cut and creased in particular ways, without separate glued-in attachments. It is remarkable the variety of effects that can be achieved under these restrictions. Chapter 1 concentrates on establishing the mathematical vocabulary and conventions we follow throughout and on applying these to parallel cuts in the cardstock. A typical application is to pop-up letters, say, "10 YEARS OLD!" In Chapter 2 we turn to the versatile V-fold and show how its motion in 3D can be understood as the intersection of three spheres, whose equations we derive and solve. We also explain the challenges of converting the "horizontal" rotary motion of the card opening to either a "vertical" rotary motion or a "flat" rotary motion. All this scare-quoted notation will of course be explained. Chapter 3 explores a simple design based on cutting parallel chords of a circle centered on the cardline, which opens to an elegant shape we call the Knight's visor. The equations describing this shape connect to beautiful nineteenth-century mathematics and—surprisingly—to caustics, shadow-shapes formed by light passing through a glass of water. In contrast to the Knight's visor, whose final, static shape is the goal, the pop-up spinner in Chapter 4 is all dynamics — amazing dynamics. The engine driving the spinner is a theorem from an undergraduate thesis on protein folding (Benbernou, 2006), which we prove in a simplified form.

The last three chapters progress to pasting-in various structures, necessarily collapsible structures to allow the card to close. Chapter 5 focuses on popping up convex polyhedra—the cube, an octahedron, etc.—and the rich mathematics surrounding polyhedra. Unlike the classical mathematics uncovered in the Knight's visor construction, the mathematics relevant to collapsing polyhedra is quite contemporary. We even quote unsolved problems (pp. 66, 99) under active investigation.

[2] cs.smith.edu/~jorourke/PopUps/

The final two chapters discuss algorithms: Chapter 6 describes an algorithm that can output instructions for a pop-up of a wide class of polyhedra, wider than the convex polyhedra considered in Chapter 5, but not as wide as “all polyhedra,” which is an as-yet unsolved problem. The final short chapter touches on the way computer scientists measure the difficulty of a problem, its “computational complexity.” Perhaps it is no surprise that deciding whether a particular pop-up structure can collapse flat is technically “intractable,” i.e., very hard!

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