

## 1

# Parallel Folds

We start with *parallel folds*, the simplest and easiest pop-up technique to understand. Despite the simplicity of parallel folds, one can reach beautiful and intricate designs with just this one construction technique, repeated. But we'll keep it simple, making our goal pop-up letters, perhaps the most common pop-up card design.

The mathematics behind parallel folds is also simple, which will provide an opportunity to set conventions and notation that will be used throughout. We will aim toward describing the 3D motion of card opening by tracking various key points on the card front and back traveling along circles in space.

## 1.1 Card Notation

Here we establish the basic notation used throughout the book, which is important because the rigor of mathematics relies on clear definitions and symbolic notation.<sup>1</sup>

A card is composed of two identical rectangles, the back  $B$  and the front  $F$ , joined and hinged along the card centerline  $L$ . (The centerline is also called the gutter or the spine.) The front and back are both rigid, usually made of stiff cardstock. We view the back as fixed to a tabletop; only the front moves. See Fig. 1.1. In general we will use uppercase letters for “big” things and lowercase letters for “small” things, for example, edges or specific points. And we will follow the mathematics tradition of using (lowercase) Greek letters for angles:  $\alpha, \beta, \theta$ , etc. In particular, we will reserve  $\theta$  (theta) to represent the card angle, the angle along the centerline between the front and the back. At  $\theta = 0^\circ$  the card is closed, at  $\theta = 90^\circ$  it is half-open, and at  $\theta = 180^\circ$  the card is fully opened flat. Often, the card is intended to only open to  $90^\circ$ , when  $F$  becomes a backdrop to the popped-up structure. This will be the case for most of the constructions in this chapter.

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<sup>1</sup>Symbols are gathered in a table on p. 121.

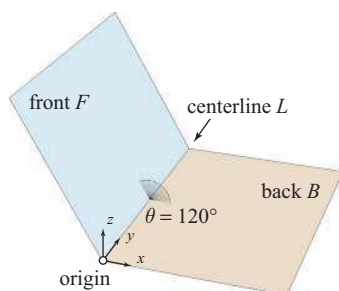


Figure 1.1 Basic card notation.

In geometry, the angle in 3D between two planes, such as  $\theta$  just described, is known as a *dihedral angle*; see Box 1.1, which relies on vectors, described in Box 1.2.

Our overall goal is to describe the geometry of the motions of the pop-up structures as the card is opened or closed. Often we reach the geometry through algebraic equations. The equations will employ variables representing points in a 3D Cartesian coordinate system, called *Cartesian* because it was introduced by René Descartes. We will consistently use  $x$  and  $y$  to represent placement in the  $xy$ -plane of the card back  $B$ , and use  $z$  as the third coordinate representing the perpendicular height above  $B$ . We place the origin of the coordinate system somewhere along the centerline  $L$ , either at the bottom end or the midpoint of  $L$ , with  $x$  increasing horizontally along the width of the back and  $y$  increasing vertically along the centerline.

### Box 1.1 Dihedral Angle

In geometry, the angle in 3D between two planes is known as a *dihedral angle*, from the Greek *di-* (two) and *hedra* (faces). A *face* is a region of a plane. Ultimately angles are measured in two dimensions, not three, between what are known as *vectors*: see Box 1.2.

In Fig. 1.2,  $F_1$  and  $F_2$  are faces sharing an edge  $e$ . The dihedral angle  $\delta$  at  $e$  can be measured by a protractor perpendicular to edge  $e$ ; so

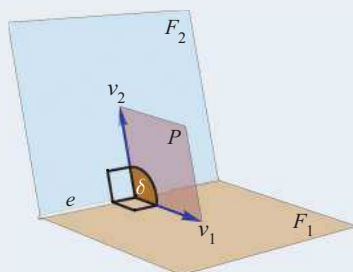


Figure 1.2 The dihedral angle  $\delta$  between faces  $F_1$  and  $F_2$ .

the protractor lies in a plane  $P$  orthogonal to  $e$ . (The term *orthogonal* is a synonym for “perpendicular.”) The vectors  $v_1$  and  $v_2$  lie along the intersection of plane  $P$  with the planes containing the faces  $F_1$  and  $F_2$ , respectively. The angle  $\delta$  measures how much  $v_2$  needs to be rotated about  $e$  to align with  $v_1$ . In this chapter,  $e$  is the card centerline  $L$  and  $\delta$  is the card angle  $\theta$ .

### Box 1.2 Vectors

Informally, a *vector* can be viewed as a directed line segment. Thus it has both a length and a direction. The two endpoints are called the *tail* and the *head*, with the head marked by an arrowhead. A vector differs from a directed line segment in that its tail is not fixed to one point of the plane or space. Rather it should be imagined to be placed anywhere, as in Fig. 1.3.

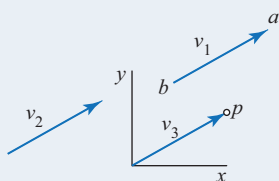


Figure 1.3 Vectors  $v_1, v_2, v_3$  are the same vector  $a - b$ .

Often it is convenient to view a point  $p$ , which has specific coordinates with respect to an origin, as a vector with its tail at the origin and its head at  $p$ . Given any two points  $a$  and  $b$ , the difference in their coordinates,  $a - b$ , is a vector, placed with the head at  $a$  and the tail at  $b$ . Here if  $a = (a_x, a_y)$  and  $b = (b_x, b_y)$ , then

$$a - b = (a_x - b_x, a_y - b_y).$$

We will make cuts and creases in the card (and in attachments to the card). Creases in the card can be either *valley* or *mountain* folds, terminology from origami with the obvious interpretation. All creases can be viewed as hinges between the flat, planar pieces to either side. Those flat pieces are often called *faces*, which are rigid except where hinged along the edges.

## 1.2 Rhombus Card

Our first pop-up is a *rhombus*, a four-sided figure (a quadrilateral) all of whose edge lengths are equal. Fig. 1.4 shows the construction from a rectangle of cardstock. After folding the cardstock in half along centerline  $L$ , the two

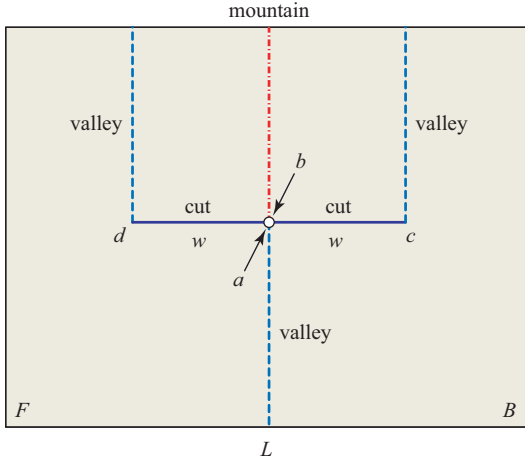


Figure 1.4 Rhombus cuts and folds, producing Fig. 1.5.

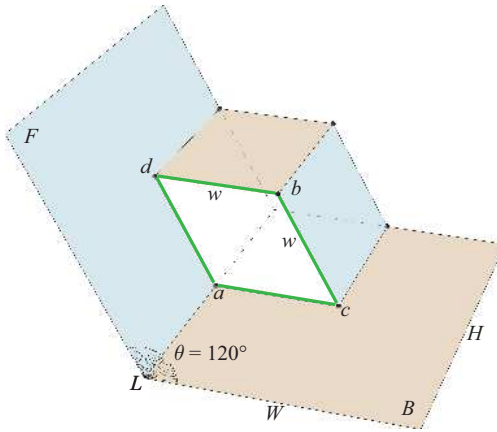


Figure 1.5 Rhombus card from the template in Fig. 1.4.

length- $w$  cuts can be achieved by one scissors cut of length  $w$  perpendicular to  $L$ . Then the mountain and valley creases shown pop out the rhombus, as illustrated in Fig. 1.5. With several points labeled, the *rim* (green in Fig. 1.5) is a 2D rhombus  $acbd$  lying in a plane orthogonal to the centerline  $L$ . The sides of the rhombus have length  $w$ . We'll use the notation  $ac$  to represent the line segment from point  $a$  to point  $c$ , and  $|ac| = w$  to indicate its length (Box 1.3.)

**Box 1.3 Segment Length  $|ab|$**

Throughout this book, we use lowercase letters to represent points, in either 2D or 3D. We often need to refer to the length of a segment

## 1.2. Rhombus Card

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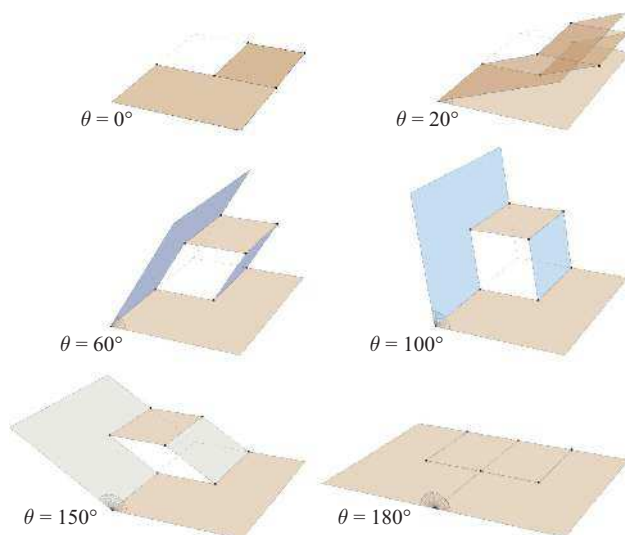


Figure 1.6 Snapshots of the rhombus card. Animation available (O'Rourke, 2021).

between two points. If  $a$  and  $b$  are two points, we use  $|ab|$  to mean the length of the line segment from  $a$  to  $b$  (which of course is the same as  $|ba|$ , the length from  $b$  to  $a$ ). In terms of 3D coordinates, with  $a = (a_x, a_y, a_z)$  and  $b = (b_x, b_y, b_z)$ ,

$$|ab| = \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2 + (a_z - b_z)^2}.$$

Often this is expressed as  $|a - b|$ , because the difference of the point coordinates is a vector from  $b$  to  $a$ , and  $|ab|$  is the length of this vector, as explained in Box 1.2. Other sources notate this as  $\|a - b\|$ , where  $\|$  indicates the *norm* of the vector.

The cut segment creates two sides, which then become the four sides of the rhombus rim. Note that in the flattened template, points  $a$  and  $b$  have the same location—they are *co-located*—and become opposite corners of the rhombus. Fig. 1.6 shows snapshots of the card at increasing values of  $\theta$  from  $0^\circ$ —completely closed—to  $180^\circ$ —fully opened, when it matches the template in Fig. 1.4.

This construction is called a *parallel fold* because the two valley creases and the one mountain crease are parallel (and parallel to the card centerline). For aesthetic reasons, often an extra layer of cardstock is pasted behind the front and back faces of the card, so that the gap formed by popping up the rhombus has a backdrop, rather than the hole shown in Fig. 1.6.

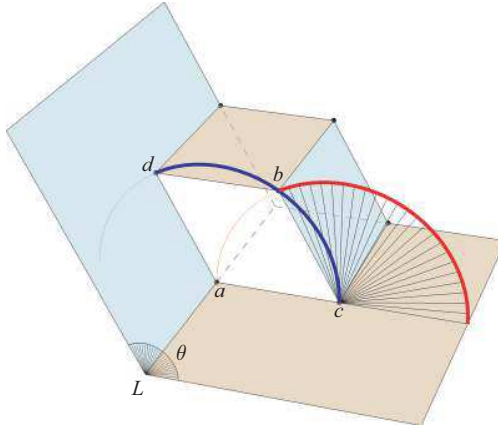


Figure 1.7 Points  $b$  and  $d$  follow circular arcs centered on  $c$  and  $a$ , respectively.

### 1.3 Circular Arc Equations

As is evident in Fig. 1.7, point  $b$  at the corner of the rhombus must remain the fixed distance  $|cb| = w$  from point  $c$  throughout the opening motion, and therefore must ride on a circle of radius  $w$  centered on  $c$ . Similarly, point  $d$  remains distance  $|ad| = w$  from  $a$ , and so follows a circle centered on  $a$ . In preparation of more complicated motions in later chapters, we now describe these motions via algebraic equations.

The equation of a radius- $w$  circle in the  $xy$ -plane, centered on the origin, is

$$x^2 + y^2 = w^2. \tag{1.1}$$

The circles followed by points  $b$  and  $d$  lie instead in the  $xz$ -plane, a plane orthogonal to the  $y$ -axis along the centerline  $L$ . Let's focus on point  $d$  revolving around  $a$ . One might think that the equation for  $d$ 's position is

$$x^2 + z^2 = w^2, \tag{1.2}$$

simply substituting  $z$  for  $y$  and otherwise not mentioning  $y$ . But because that equation does not constrain  $y$ , it describes circles for all values of  $y$ . So Eq. 1.2 describes an infinite cylinder centered on the  $y$ -axis (and in a 3D coordinate system, Eq. 1.1 denotes an infinite cylinder centered on the  $z$ -axis). To describe the circle that  $d$  follows, we need two equations, including one pinning down the precise  $y$ -coordinate for point  $a$ .

Let us say that the card back  $B$  has width  $W$  and height  $H$  (uppercase distinguishing from lowercase  $w$  and  $h$ ). Then point  $a$  has coordinates  $a = (0, \frac{H}{2}, 0)$ , and  $d$  is constrained by these two equations “simultaneously”:

$$\begin{aligned} x^2 + z^2 &= w^2, \\ y &= \frac{H}{2}. \end{aligned}$$

### 1.3. Circular Arc Equations

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We will frequently need to “solve” equations simultaneously, but here there is nothing to solve: we are just specifying that both equations constrain  $x, y, z$  at the same time.

Similarly, point  $b$  follows the circle arc centered on  $c = (w, \frac{H}{2}, 0)$  defined by these two equations, where  $x$  is shifted by  $w$ :

$$\begin{aligned}(x - w)^2 + z^2 &= w^2, \\ y &= \frac{H}{2}.\end{aligned}$$

Lastly we presage *parametric equations*, which were used to draw the spokes in Fig. 1.7 and which will play a significant role in later chapters (Box 2.2). Parametric equations rely on a *parameter*, which in our situation is the card angle  $\theta$ . The equations we just derived describe the full circle for point  $b$  “all at once.” Parametric equations pinpoint the location of  $b$  for a specific value of  $\theta$ :

$$\begin{aligned}x &= w + w \cos \theta, \\ y &= \frac{H}{2}, \\ z &= w \sin \theta.\end{aligned}\tag{1.3}$$

For example, if  $\theta = 90^\circ$ , then  $\cos \theta$  and  $\sin \theta$  are 0 and 1, respectively, so  $(x, y, z) = (w, \frac{H}{2}, w)$ , which places  $b$  directly above  $c = (w, \frac{H}{2}, 0)$ . (For more on cosine and sine, see Box 1.4.)

#### Box 1.4 Trigonometry

We will use trigonometry sparingly and never need more than the basic relationships illustrated in Fig. 1.8, always measuring angles counter-clockwise. Note that not only is  $\tan \theta$  the ratio of the altitude to the base of the triangle, but it is the length of the tangent from  $p$  to the  $x$ -axis. This latter relationship helps to understand why, as  $\theta$  approaches  $90^\circ$ ,  $\tan \theta$  grows without bound—approaches infinity.

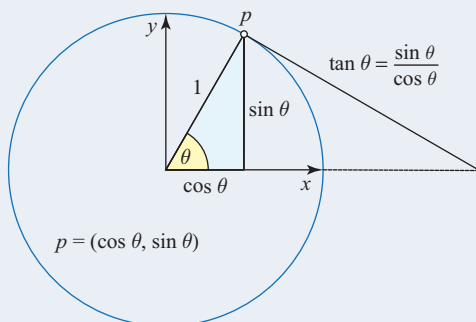


Figure 1.8 Basic trigonometry: unit-radius circle.

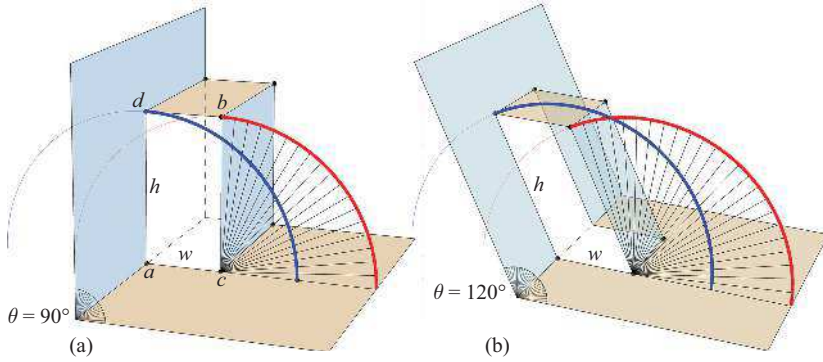


Figure 1.9 (a) Asymmetric cut:  $h > w$ . (b) Two parallel asymmetric cuts.

## 1.4 Parallelograms

Two simple modifications of the rhombus construction turn it into a useful pop-up structure. First, rather than cut a length  $w$  to each side of the centerline  $L$ , an asymmetric cut of  $w$  on the back  $B$  and  $h$  on the front  $F$  leads to a pop-up parallelogram,  $w \times h$  at card angle  $\theta = 90^\circ$ . See Fig. 1.9(a).

Second, two identical parallel cuts, symmetric or asymmetric about  $L$ , result in a centered pop-up, rather than the parallelogram extending to the card's top edge. See Fig. 1.9(b). With  $h > w$  as in the figure, the pop-up can serve as a face into which letters can be carved. When  $h < w$ , the pop-up often serves as a platform on which other structures are built.

### Exercise 1.1 Practice: Within Card Profile

With the card width  $W$  and height  $H$ , which combinations of the pop-up parallelogram dimensions  $w$  and  $h$  ensure that no portion of the pop-up “sticks out” when the card is fully closed at  $\theta = 0^\circ$ ? (See Fig. 1.6.)

### Exercise 1.2 Understanding: Arcs Intersection

In Fig. 1.9(b), what are the coordinates of the point  $p$  at which the red and blue arcs cross? Here red marks the trajectory of point  $b$ , and blue that of point  $d$ .



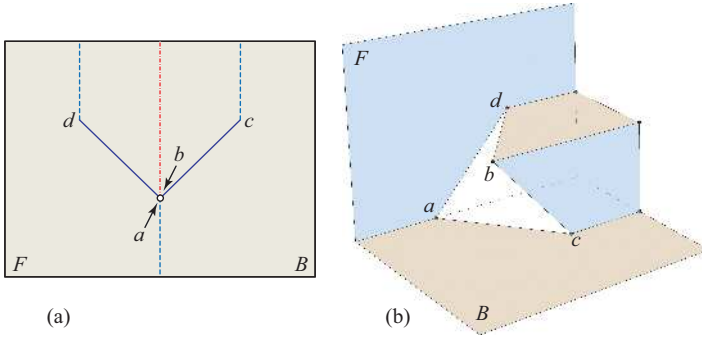


Figure 1.10 (a) Template for angled cuts. (b) The pop-up.

## 1.5 Cut Variations

We've examined symmetric and asymmetric cuts, but many other variations are possible. For example, angled but symmetric cuts (and still creases parallel to  $L$ ), as in Fig. 1.10(a), lead to a very similar pop-up with parallel creases, as in Fig. 1.10(b).

Before we turn to popping up letters, the reader might enjoy exploring several other variants in the following exercises, perhaps verifying understanding with scissors and paper (cardstock is not essential for quick experiments).

### Exercise 1.3 Understanding: Parallel Unequal

Describe the behavior of the pop-up whose template is shown in Fig. 1.11.

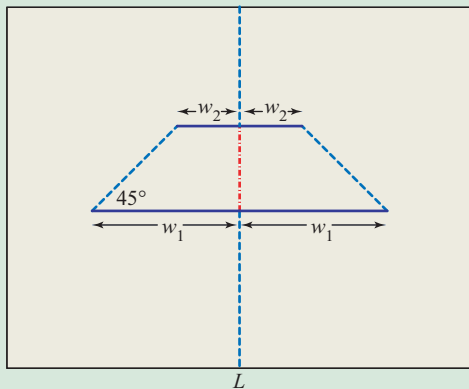


Figure 1.11 Cuts are centered on  $L$ , parallel, but of unequal lengths.

### Exercise 1.4 Understanding: Three Parallel Cuts

Describe the behavior of the pop-up whose template is shown in Fig. 1.12.

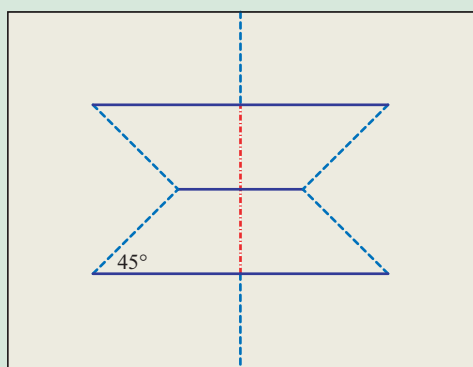


Figure 1.12 Three centered, parallel cuts.

### Exercise 1.5 Understanding: Slanted Parallel

Describe the behavior of the pop-up whose template is shown in Fig. 1.13, with valley creases as shown.

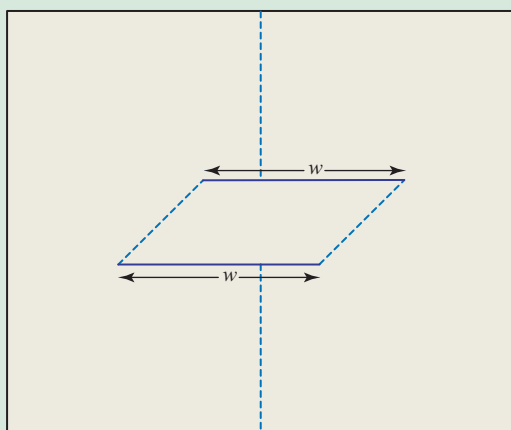


Figure 1.13 Parallel but slanted cuts.

## 1.6 Pop-Up Letters

It is now a short step from popping up a parallelogram box to popping up letters “carved” into the front of such a box. Fig. 1.14 illustrates the idea for the letter