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Basics

Topology, mainly algebraic topology, is the fundamental mathematical subject on which topological data analysis is based. In this chapter, we introduce some of the very basics of this subject that are used in this book. First, in Section 1.1, we give the definition of a topological space and other notions such as open and closed sets, covers, and subspace topology that are derived from it. These notions are quite abstract in the sense that they do not require any geometry. However, the intuition of topology becomes more concrete to nonmathematicians when we bring geometry into the mix. Section 1.2 is devoted to make the connection between topology and geometry through what is called metric spaces.

Maps such as homeomorphism and homotopy equivalence play a significant role to relate topological spaces. They are introduced in Section 1.3. At the heart of these definitions sits the important notion of continuous functions which generalizes the concept mainly known for Euclidean domains to topological spaces. Certain categories of topological spaces become important for their wide presence in applications. Manifolds are one such category which we introduce in Section 1.4. Functions on them satisfying certain conditions are presented in Section 1.5. They are well known as Morse functions. The critical points of such functions relate to the topology of the manifold they are defined on. We introduce these concepts in the smooth setting in this chapter, and later adapt them for the piecewise-linear domains that are amenable to finite computations.

1.1 Topological Space

The basic object in a topological space is a ground set whose elements are called points. A topology on these points specifies how they are *connected* by listing what points constitute a neighborhood – the so-called *open set*.

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The expression "rubber-sheet topology" commonly associated with the term "topology" exemplifies this idea of connectivity of neighborhoods. If we bend and stretch a sheet of rubber, it changes shape but always preserves the neighborhoods in terms of the points and how they are connected.

We first introduce basic notions from point set topology. These notions are prerequisites for more sophisticated topological ideas – manifolds, homeomorphism, isotopy, and other maps – used later to study algorithms for topological data analysis. Homeomorphisms, for example, offer a rigorous way to state that an operation preserves the topology of a domain, and isotopy offers a rigorous way to state that the domain can be deformed into a shape without ever colliding with itself.

Perhaps it is more intuitive to understand the concept of topology in the presence of a metric because then we can use the metric balls such as Euclidean balls in a Euclidean space to define neighborhoods – the open sets. Topological spaces provide a way to abstract out this idea without a metric or point coordinates, so they are more general than metric spaces. In place of a metric, we encode the connectivity of a point set by supplying a list of all of the open sets. This list is called a *system* of subsets of the point set. The point set and its system together describe a topological space.

Definition 1.1. (Topological space) A *topological space* is a point set $\mathbb{T}$ endowed with a *system of subsets* T, which is a set of subsets of $\mathbb{T}$ that satisfies the following conditions:

- $\emptyset, \mathbb{T} \in T$.
- For every $U \subseteq T$, the union of the subsets in U is in T.
- For every finite $U \subseteq T$, the common intersection of the subsets in U is in T.

The system T is called a *topology* on $\mathbb{T}$. The sets in T are called the *open sets* in $\mathbb{T}$. A *neighborhood* of a point $p \in \mathbb{T}$ is an open set containing p.

First, we give examples of topological spaces to illustrate the definition above. These examples have the set $\mathbb{T}$ finite.

Example 1.1. *Let* $\mathbb{T} = \{0, 1, 3, 5, 7\}$. *Then,* $T = \{\emptyset, \{0\}, \{1\}, \{5\}, \{1, 5\}, \{0, 1\}, \{0, 1, 5\}, \{0, 1, 3, 5, 7\}\}$ *is a topology because* $\emptyset$ *and* $\mathbb{T}$ *are in* T *as required by the first axiom, the union of any sets in* T *is in* T *as required by the second axiom, and the intersection of any two sets is also in* T *as required by the third axiom. However,* $T = \{\emptyset, \{0\}, \{1\}, \{1, 5\}, \{0, 1, 5\}, \{0, 1, 3, 5, 7\}\}$ *is not a topology because the set* $\{0, 1\} = \{0\} \cup \{1\}$ *is missing.*

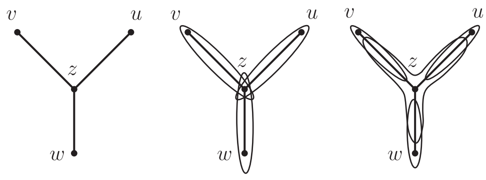

Figure 1.1 Example 1.3: (a) a graph as a topological space, stars of the vertices and edges as open sets; (b) a closed cover with three elements; and (c) an open cover with four elements.

Example 1.2. *Let* $\mathbb{T} = \{u, v, w\}$. *The power set* $2^{\mathbb{T}} = \{\emptyset, \{u\}, \{v\}, \{w\}, \{u, v\}, \{u, w\}, \{v, w\}, \{u, v, w\}\}$ *is a topology. For any ground set* $\mathbb{T}$*, the power set is always a topology on it which is called the* discrete topology.

One may take a subset of the power set as a ground set and define a topology, as the next example shows. We will recognize later that the ground set here corresponds to simplices in a simplicial complex and the "stars" of simplices generate all open sets of a topology.

Example 1.3. *Let* $\mathbb{T} = \{u, v, w, z, (u, z), (v, z), (w, z)\}$*; this can be viewed as a graph with four vertices and three edges as shown in Figure 1.1. Let*

- $T_1 = \{\{(u, z)\}, \{(v, z)\}, \{(w, z)\}\}$ *and*
- $T_2 = \{\{(u, z), u\}, \{(v, z), v\}, \{(w, z), w\}, \{(u, z), (v, z), (w, z), z\}\}$.

Then, $T = 2^{T_1 \cup T_2}$ *is a topology because it satisfies all three axioms. All open sets of* T *are generated by the union of elements in* $B = T_1 \cup T_2$ *and there is no smaller set with this property. Such a set* B *is called a basis of* T*. We will see later in the next chapter (Section 2.1) that these are open* stars *of all vertices and edges.*

We now present some more definitions that will be useful later.

Definition 1.2. (Closure; Closed sets) A set Q is *closed* if its complement $\mathbb{T} \setminus Q$ is open. The *closure* $\mathrm{Cl}\, Q$ of a set $Q \subseteq T$ is the smallest closed set containing Q.

In Example 1.1, the set $\{3, 5, 7\}$ is closed because its complement $\{0, 1\}$ in $\mathbb{T}$ is open. The closure of the open set $\{0\}$ is $\{0, 3, 7\}$ because it is the smallest closed set (complement of open set $\{1, 5\}$) containing 0. In Example 1.2, all sets are both open and closed. In Example 1.3, the set $\{u, z, (u, z)\}$ is closed, but the set $\{z, (u, z)\}$ is neither open nor closed. Interestingly, observe that

$\{z\}$ is closed. The closure of the open set $\{u, (u, z)\}$ is $\{u, z, (u, z)\}$. In all examples, the sets $\varnothing$ and $\mathbb{T}$ are both open and closed.

Definition 1.3. Given a topological space $(\mathbb{T}, T)$, the *interior* Int A of a subset $A \subseteq \mathbb{T}$ is the union of all open subsets of A. The *boundary* of A is Bd $A = \text{Cl}\, A \setminus \text{Int}\, A$.

The interior of the set $\{3, 5, 7\}$ in Example 1.1 is $\{5\}$ and its boundary is $\{3, 7\}$.

Definition 1.4. (Subspace topology) For every point set $\mathbb{U} \subseteq \mathbb{T}$, the topology T induces a *subspace topology* on $\mathbb{U}$, namely the system of open subsets $U = \{P \cap \mathbb{U} \colon P \in T\}$. The point set $\mathbb{U}$ endowed with the system U is said to be a *topological subspace* of $\mathbb{T}$.

In Example 1.1, consider the subset $\mathbb{U} = \{1, 5, 7\}$. It has the subspace topology

$$U = \{\varnothing, \{1\}, \{5\}, \{1, 5\}, \{1, 5, 7\}\}.$$

In Example 1.3, the subset $\mathbb{U} = \{u, (u, z), (v, z)\}$ has the subspace topology

$$\{\varnothing, \{u, (u, z)\}, \{(u, z)\}, \{(v, z)\}, \{(u, z), (v, z)\}, \{u, (u, z), (v, z)\}\}.$$

Definition 1.5. (Connected) A topological space $(\mathbb{T}, T)$ is *disconnected* if there are two disjoint non-empty open sets $U, V \in T$ so that $\mathbb{T} = U \cup V$. A topological space is *connected* if it is not disconnected.

The topological space in Example 1.1 is connected. However, the topological subspace (Definition 1.4) induced by the subset $\{0, 1, 5\}$ is disconnected because it can be obtained as the union of two disjoint open sets $\{0, 1\}$ and $\{5\}$. The topological space in Example 1.3 is also connected, but the subspace induced by the subset $\{(u, z), (v, z), (w, z)\}$ is disconnected.

Definition 1.6. (Cover; Compact) An *open (closed) cover* of a topological space $(\mathbb{T}, T)$ is a collection C of open (closed) sets so that $\mathbb{T} = \bigcup_{c \in C} c$. The topological space $(\mathbb{T}, T)$ is called *compact* if every open cover C of it has a finite *subcover*, that is, there exists $C' \subseteq C$ such that $\mathbb{T} = \bigcup_{c \in C'} c$ and C' is finite.

In Figure 1.1(b), the cover consisting of $\{\{u, z, (u, z)\}, \{v, z, (v, z)\}, \{w, z, (w, z)\}\}$ is a closed cover whereas the cover consisting of $\{\{u, (u, z)\}, \{v, (v, z)\},$

$\{w, (w, z)\}, \{z, (u, z), (v, z), (w.z)\}\}$ in Figure 1.1(c) is an open cover. Any topological space with finite point set $\mathbb{T}$ is compact because all of its covers are finite. Thus, all topological spaces in the discussed examples are compact. We will see examples of noncompact topological spaces where the ground set is infinite.

In the above examples, the ground set $\mathbb{T}$ is finite. It can be infinite in general and a topology may have uncountably infinitely many open sets containing uncountably infinitely many points.

Next, we introduce the concept of *quotient topology*. Given a space $(\mathbb{T}, T)$ and an equivalence relation $\sim$ on elements in $\mathbb{T}$, one can define a topology induced by the original topology T on the quotient set $\mathbb{T}/\sim$ whose elements are equivalence classes $[x]$ for every point $x \in \mathbb{T}$.

Definition 1.7. (Quotient topology) Given a topological space $(\mathbb{T}, T)$ and an equivalence relation $\sim$ defined on the set $\mathbb{T}$, a quotient space $(\mathbb{S}, S)$ induced by $\sim$ is defined by the set $\mathbb{S} = \mathbb{T}/\sim$ and the *quotient topology* S where

$$S := \big\{U \subseteq \mathbb{S} \mid \{x \colon [x] \in U\} \in T\big\}.$$

We will see the use of quotient topology in Chapter 7 when we study Reeb graphs.

Infinite topological spaces may seem baffling from a computational point of view, because they may have uncountably infinitely many open sets containing uncountably infinitely many points. The easiest way to define such a topological space is to inherit the open sets from a metric space. A topology on a metric space excludes information that is not topologically essential. For instance, the act of stretching a rubber sheet changes the distances between points and thereby changes the metric, but it does not change the open sets or the topology of the rubber sheet. In the next section, we construct such a topology on a metric space and examine it from the concept of limit points.

1.2 Metric Space Topology

Metric spaces are a special type of topological space commonly encountered in practice. Such a space admits a *metric* that specifies the scalar *distance* between every pair of points satisfying certain axioms.

Definition 1.8. (Metric space) A *metric space* is a pair $(\mathbb{T}, \mathsf{d})$ where $\mathbb{T}$ is a set and d is a distance function $\mathsf{d} \colon \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ satisfying the following properties:

- $\mathsf{d}(p, q) = 0$ if and only if $p = q$ for all $p \in \mathbb{T}$;
- $\mathsf{d}(p, q) = \mathsf{d}(q, p)$ for all $p, q \in \mathbb{T}$;
- $\mathsf{d}(p, q) \le \mathsf{d}(p, r) + \mathsf{d}(r, q)$ for all $p, q, r \in \mathbb{T}$.

It can be shown that the three axioms above imply that $\mathsf{d}(p, q) \ge 0$ for every pair $p, q \in \mathbb{T}$. In a metric space $\mathbb{T}$, an open *metric ball* with center c and radius r is defined to be the point set $B_o(c, r) = \{p \in \mathbb{T} : \mathsf{d}(p, c) < r\}$. Metric balls define a topology on a metric space.

Definition 1.9. (Metric space topology) Given a metric space $\mathbb{T}$, all metric balls $\{B_o(c, r) \mid c \in \mathbb{T} \text{ and } 0 < r \le \infty\}$ and their union constituting the open sets define a topology on $\mathbb{T}$.

All definitions for general topological spaces apply to metric spaces with the above defined topology. However, we give alternative definitions using the concept of limit points which may be more intuitive.

As we have mentioned already, the heart of topology is the question of what it means for a set of points to be *connected*. After all, two distinct points cannot be adjacent to each other; they can only be connected to one another by passing through uncountably many intermediate points. The idea of *limit points* helps express this concept more concretely, specifically in the case of metric spaces.

We use the notation $\mathsf{d}(\cdot, \cdot)$ to express minimum distances between point sets $P, Q \subseteq \mathbb{T}$:

$$\mathsf{d}(p, Q) = \inf\{\mathsf{d}(p, q) : q \in Q\},$$
$$\mathsf{d}(P, Q) = \inf\{\mathsf{d}(p, q) : p \in P, q \in Q\}.$$

Definition 1.10. (Limit point) Let $Q \subseteq \mathbb{T}$ be a point set. A point $p \in \mathbb{T}$ is a *limit point* of Q, also known as an *accumulation point* of Q, if for every real number $\epsilon > 0$, however tiny, Q contains a point $q \ne p$ such that $d(p, q) < \epsilon$.

In other words, there is an infinite sequence of points in Q that gets successively closer and closer to p – without actually being p – and gets arbitrarily close. Stated succinctly, $\mathsf{d}(p, Q \setminus \{p\}) = 0$. Observe that it does not matter whether $p \in Q$ or not.

To see the parallel between the definitions given in this subsection and the definitions given before, it is instructive to define limit points also for general topological spaces. In particular, a point $p \in \mathbb{T}$ is a limit point of a set $Q \subseteq \mathbb{T}$ if every open set containing p intersects Q.

Definition 1.11. (Connected) A point set $Q \subseteq \mathbb{T}$ is called *disconnected* if Q can be partitioned into two disjoint non-empty sets U and V so that there is no

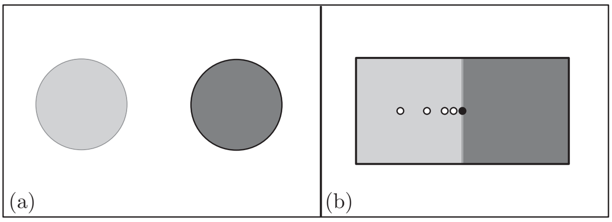

Figure 1.2 (a) The point set is disconnected; it can be partitioned into two connected subsets shaded differently. (b) The point set is connected; the black point at the center is a limit point of the points shaded lightly.

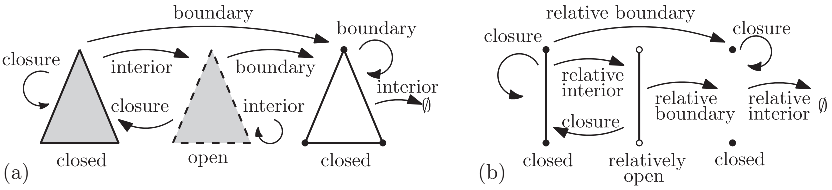

Figure 1.3 Closed, open, and relatively open point sets in the plane. Dashed edges and open circles indicate points missing from the point set.

point in U that is a limit point of V, and no point in V that is a limit point of U. (See Figure 1.2[a] for an example.) If no such partition exists, Q is *connected*, like the point set in Figure 1.2(b).

We can also distinguish between closed and open point sets using the concept of limit points. Informally, a triangle in the plane is *closed* if it contains all the points on its edges, and *open* if it excludes all the points on its edges, as illustrated in Figure 1.3. The idea can be formally extended to any point set.

Definition 1.12. (Closure; Closed; Open) The *closure* of a point set $Q \subseteq \mathbb{T}$, denoted $\mathrm{Cl}\, Q$, is the set containing every point in Q and every limit point of Q. A point set Q is *closed* if $Q = \mathrm{Cl}\, Q$, that is, Q contains all its limit points. The *complement* of a point set Q is $\mathbb{T} \setminus Q$. A point set Q is *open* if its complement is closed, that is, $\mathbb{T} \setminus Q = \mathrm{Cl}\,(\mathbb{T} \setminus Q)$.

For example, consider the open interval $(0, 1) \subset \mathbb{R}$, which contains every $r \in \mathbb{R}$ so that $0 < r < 1$. Let $[0, 1]$ denote a *closed interval* $(0, 1) \cup \{0\} \cup \{1\}$. The numbers 0 and 1 are both limit points of the open interval, so $\mathrm{Cl}\,(0, 1) = [0, 1] = \mathrm{Cl}\,[0, 1]$. Therefore, $[0, 1]$ is closed and $(0, 1)$ is not. The numbers 0 and 1 are also limit points of the complement of the closed interval, $\mathbb{R} \setminus [0, 1]$, so $(0, 1)$ is open, but $[0, 1]$ is not.

The definition of *open set* of course depends on the space being considered. A triangle τ that is missing the points on its edges is open in the two-dimensional affine Euclidean space supporting τ. However, it is not open in the Euclidean space $\mathbb{R}^3$. Indeed, every point in τ is a limit point of $\mathbb{R}^3 \setminus \tau$, because we can find sequences of points that approach τ from the side. In recognition of this caveat, a simplex $\sigma \subset \mathbb{R}^d$ is said to be *relatively open* if it is open relative to its affine hull. Figure 1.3 illustrates this fact where, in this example, the metric space is $\mathbb{R}^2$.

We can define the interior and boundary of a set using the notion of limit points also. Informally, the boundary of a point set Q is the set of points where Q meets its complement $\mathbb{T} \setminus Q$. The interior of Q contains all the other points of Q.

Definition 1.13. (Boundary; Interior) The *boundary* of a point set Q in a metric space $\mathbb{T}$, denoted $\mathrm{Bd}\,Q$, is the intersection of the closures of Q and its complement; that is, $\mathrm{Bd}\,Q = \mathrm{Cl}\,Q \cap \mathrm{Cl}\,(\mathbb{T} \setminus Q)$. The *interior* of Q, denoted $\mathrm{Int}\,Q$, is $Q \setminus \mathrm{Bd}\,Q = Q \setminus \mathrm{Cl}\,(\mathbb{T} \setminus Q)$.

For example, $\mathrm{Bd}\,[0, 1] = \{0, 1\} = \mathrm{Bd}\,(0, 1)$ and $\mathrm{Int}\,[0, 1] = (0, 1) = \mathrm{Int}\,(0, 1)$. The boundary of a triangle (closed or open) in the Euclidean plane is the union of the triangle's three edges, and its interior is an open triangle, illustrated in Figure 1.3. The terms *boundary* and *interior* have similar subtlety as open sets: the boundary of a triangle embedded in $\mathbb{R}^3$ is the whole triangle, and its interior is the empty set. However, relative to its affine hull, its interior and boundary are defined exactly as in the case of triangles embedded in the Euclidean plane. Interested readers can draw the analogy between this observation and the definition of interior and boundary of a manifold that appear later in Definition 1.23.

We have seen a definition of the compactness of a point set in a topological space (Definition 1.6). We define it differently here for a metric space. It can be shown that the two definitions are equivalent.

Definition 1.14. (Bounded; Compact) The *diameter* of a point set Q is $\sup_{p,q \in Q} \mathsf{d}(p, q)$. The set Q is *bounded* if its diameter is finite, and is *unbounded* otherwise. A point set Q in a metric space is *compact* if it is closed and bounded.

In the Euclidean space $\mathbb{R}^d$ we can use the standard Euclidean distance as the choice of metric. On the surface of a coffee mug, we could choose the Euclidean distance too; alternatively, we could choose the *geodesic distance*, namely the length of the shortest path from p to q on the mug's surface.

Example 1.4. (Euclidean ball) *In $\mathbb{R}^d$, the Euclidean d-ball with center c and radius r, denoted $B(c,r)$, is the point set $B(c,r) = \{p \in \mathbb{R}^d : \mathsf{d}(p,c) \leq r\}$. A 1-ball is an edge, and a 2-ball is called a disk. A unit ball is a ball with radius 1. The boundary of the d-ball is called the Euclidean $(d-1)$-sphere and denoted $S(c,r) = \{p \in \mathbb{R}^d : \mathsf{d}(p,c) = r\}$. The name expresses the fact that we consider it a $(d-1)$-dimensional point set – to be precise, a $(d-1)$-dimensional manifold – even though it is embedded in d-dimensional space. For example, a circle is a 1-sphere, and a layman's "sphere" in $\mathbb{R}^3$ is a 2-sphere. If we remove the boundary from a ball, we have the open Euclidean d-ball $B_o(c,r) = \{p \in \mathbb{R}^d : \mathsf{d}(p,c) < r\}$.*

The topological spaces that are subspaces of a metric space such as $\mathbb{R}^d$ inherit their topology as a subspace topology. Examples of topological subspaces are the Euclidean d-ball $\mathbb{B}^d$, Euclidean d-sphere $\mathbb{S}^d$, open Euclidean d-ball $\mathbb{B}^d_o$, and Euclidean half-ball $\mathbb{H}^d$, where

$$\begin{aligned}
\mathbb{B}^d &= \{x \in \mathbb{R}^d : \|x\| \leq 1\},\\
\mathbb{S}^d &= \{x \in \mathbb{R}^{d+1} : \|x\| = 1\},\\
\mathbb{B}^d_o &= \{x \in \mathbb{R}^d : \|x\| < 1\},\\
\mathbb{H}^d &= \{x \in \mathbb{R}^d : \|x\| < 1 \text{ and } x_d \geq 0\}.
\end{aligned}$$

1.3 Maps, Homeomorphisms, and Homotopies

The equivalence of two topological spaces is determined by how the points that comprise them are connected. For example, the surface of a cube can be deformed into a sphere without cutting or gluing it because they are connected the same way. They have the same topology. This notion of topological equivalence can be formalized via functions that send the points of one space to points of the other while preserving the connectivity.

This preservation of connectivity is achieved by preserving the open sets. A function from one space to another that preserves the open sets is called a *continuous function* or a *map*. Continuity is a vehicle to define topological equivalence, because a continuous function can send many points to a single point in the target space, or send no points to a given point in the target space. If the former does not happen, that is, when the function is injective, we call it an *embedding* of the domain into the target space. True equivalence is given by a *homeomorphism*, a bijective function from one space to another which has continuity as well as a continuous inverse. This ensures that open sets are preserved in both directions.

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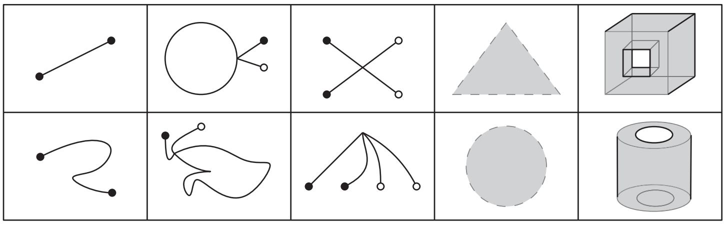

Figure 1.4 Each point set in this figure is homeomorphic to the point set above or below it, but not to any of the others. Open circles indicate points missing from the point set, as do the dashed edges in the point sets second from the right.

Definition 1.15. (Continuous function; Map) A function $f : \mathbb{T} \to \mathbb{U}$ from the topological space $\mathbb{T}$ to another topological space $\mathbb{U}$ is *continuous* if for every open set $Q \subseteq \mathbb{U}$, $f^{-1}(Q)$ is open. Continuous functions are also called *maps*.

Definition 1.16. (Embedding) A map $g : \mathbb{T} \to \mathbb{U}$ is an *embedding* of $\mathbb{T}$ into $\mathbb{U}$ if g is injective.

A topological space can be *embedded* into a Euclidean space by assigning coordinates to its points so that the assignment is continuous and injective. For example, drawing a triangle on paper is an embedding of $\mathbb{S}^1$ into $\mathbb{R}^2$. There are topological spaces that cannot be embedded into a Euclidean space, or even into a metric space – these spaces cannot be represented by any metric.

Next we define a homeomorphism that connects two spaces that have essentially the same topology.

Definition 1.17. (Homeomorphism) Let $\mathbb{T}$ and $\mathbb{U}$ be topological spaces. A *homeomorphism* is a bijective map $h : \mathbb{T} \to \mathbb{U}$ whose inverse is continuous too.

Two topological spaces are *homeomorphic* if there exists a homeomorphism between them.

Homeomorphism induces an equivalence relation among topological spaces, which is why two homeomorphic topological spaces are called *topologically equivalent*. Figure 1.4 shows pairs of homeomorphic topological spaces. A less obvious example is that the open d-ball $\mathbb{B}_o^d$ is homeomorphic to the Euclidean space $\mathbb{R}^d$, given by the homeomorphism $h(x) = x/(1 - ||x||)$. The same map also exhibits that the half-ball $\mathbb{H}^d$ is homeomorphic to the Euclidean half-space $\{x \in \mathbb{R}^d : x_d \geq 0\}$.

For maps between compact spaces, there is a weaker condition to be verified for homeomorphisms because of the following property.