1 Introduction

Science cannot do without mathematics, so if you believe science, you had better believe mathematics. That, in lay terms, is the Indispensability Argument. The striking thing about this argument, which we will set out more formally shortly, is that it locates the justification for mathematics outside mathematics itself. Scientists develop theories about the physical world as varied as general relativity, the atomic theory of matter, Darwin's theory of evolution by natural selection, and many more. The best of these explain a wide range of empirical phenomena, make accurate predictions, and are widely believed. But not only that, goes the claim: to the extent that they use mathematics, such theories give us a reason for believing its truth. And even a cursory look at scientific theories shows how many of them use mathematics and how extensively. So we should believe mathematics.

Take the example of Fermat's Last Theorem: if positive integers n, x, y, and z are such that $x^n + y^n = z^n$ then $n = 1$ or $n = 2$. Following its proof by Andrew Wiles and Richard Taylor in the mid-1990s, mathematicians now accept the theorem as true. Indispensabilists maintain that we are justified in believing Fermat's Last Theorem because the axioms and rules needed to prove it are all justified by their utility to science. Science either needs these axioms and rules directly or needs mathematical claims best systematized in terms of them. Mathematics is indispensable to science because science cannot manage without it.

The Indispensability Argument is regularly said to be the strongest argument for believing in the truth of mathematics. And of course most philosophers, not to mention the overwhelming majority of mathematicians and laypeople, take mathematics to be true. In fact, even some philosophers who think (nonvacuous) mathematical claims are not true regard the Indispensability Argument as the main argument worth taking seriously. An example is Hartry Field, who in the preface to his book *Science Without Numbers* declares: "The only non-question-begging arguments I have ever heard for the view that mathematics is a body of truths all rest ultimately on the applicability of mathematics to the physical world; so if applicability to the physical world isn't a good argument either, then there is no reason to regard any part of mathematics as true" (Field 1980, p. viii).

The Indispensability Argument – in its present and classic version that applies to mathematics – is often called the "Quine–Putnam Indispensability Argument," after the Harvard philosophers W. V. Quine and Hilary Putnam. No exact formulation of the argument can be found in Quine's works, though loose versions of the idea certainly appear in them from the early 1950s onwards.

Putnam (1971) articulates the argument more precisely in his *Philosophy of Logic*.

The Indispensability Argument aims to establish the truth of mathematical claims, and thereby the existence of mathematical objects. One could question the inference from the truth of mathematical claims to the existence of mathematical objects, as Putnam himself did in a later incarnation, but we shall not do so here.[1] Following Colyvan (2001, p. 1), but with one small difference (see Footnote 7), we favor a two-premise version of the argument:[2]

The Indispensability Argument

1. We ought rationally to be ontologically committed to objects indispensable to our best scientific theories.
2. Mathematical objects are indispensable to our best scientific theories.

We ought rationally to be ontologically committed to mathematical objects.

Mathematical objects include numbers, sets, functions, groups, and the like. It is generally believed that if they exist, these objects are *abstract*, meaning, roughly, nonspatiotemporal and noncausal. In fact, mathematical objects serve as paradigms of abstract objects. Those who believe mathematical objects exist and are abstract are known as *platonists*. Platonists about a branch of mathematics take its accepted statements to be meaningful, declarative, and true, and construe them at face value as being about abstract objects. Platonists about arithmetic, for example, understand "11 is prime" as the claim that the abstract object 11 has the property of being prime. If we assume that mathematical objects are abstract,[3] the Indispensability Argument is then an argument for platonism.

Quine initially rejected platonism, and in fact believed no abstract objects exist, but later became a platonist on indispensability grounds. Early on in his career, he tried to regiment the mathematical parts of science so as to avoid commitment to mathematical, and thus abstract, objects. (He assumed that if mathematics contains

1 See Paseau (2007) and chapter 7 of Paseau (forthcoming b). For Putnam, see Footnote 6. Pincock (2012, chapter 9) mentions other writers who have questioned this inference.

2 Later, from Section 5 onwards, when there is another version of the argument to contrast it with, this original version of the argument will be known as the Quine–Putnam Indispensability Argument.

3 This assumption is very common but not universal: see, for example, the "Aristotelian realism" defended in Franklin (2014).

reference to objects then they must be abstract.) He wrote a famous article with Nelson Goodman that sought to do this for some elementary portions of applied mathematics (Quine and Goodman 1947). For example, the statement

$$2 + 3 = 5$$

can be paraphrased as the logical truth

$$(\exists_2 x(Fx) \wedge \exists_3 x(Gx) \wedge \forall x \neg(Fx \wedge Gx)) \rightarrow \exists_5 x(Fx \vee Gx),$$

which may be read as "If there are exactly two Fs and exactly three Gs and nothing is both F and G then there are exactly five F-or-Gs." Here "$\exists_2 x(Fx)$" abbreviates

$$\exists x \exists y(Fx \wedge Fy \wedge x \neq y \wedge \forall z(Fz \rightarrow (z = x \vee z = y))),$$

which involves no reference to the number 2. Similarly for $\exists_3 x(Fx)$ and $\exists_5 x(Fx)$. Since the paraphrases contain no quantifiers ranging over a domain of abstract objects, nor singular terms denoting abstract objects, by Quine's lights, they harbor no commitment to such objects.[4]

And yet despite his best efforts, Quine came to the conclusion that this approach could not be made to work for nonelementary parts of mathematics that go far beyond such simple claims as $2 + 3 = 5$. Try as he might, he could not avoid reference to, or quantification over, mathematical objects. His apparent failure convinced him that it could not be done at all. With initial reluctance, he grasped the nettle and embraced the Indispensability Argument's second premise (in our terminology). He recanted his earlier wholesale rejection of the abstract and accepted abstract mathematical objects.[5]

Quine did not believe that the meanings of our scientific beliefs rigidly constrain regimentation. In this respect, he differed from Putnam, who was more interested in respecting the meaning of what "one daily presupposes" in the practice of science. The latter wrote:

> quantification over mathematical entities is indispensable for science, both formal and physical; therefore we should accept such quantification; but this commits us to accepting the existence of the mathematical entities in question. This type of argument stems, of course, from Quine, who has for years stressed both the indispensability of quantification over mathematical entities and the intellectual dishonesty of denying the existence of what one daily presupposes. (Putnam 1979, p. 347)

[4] Along with virtually all contemporary philosophers, we will assume throughout, mostly implicitly, something like these criteria of ontological commitment.

[5] But not any abstract objects beyond the mathematical. For example, he continued to reject meanings and propositions.

Putnam here states that the argument he puts forward "stems" from Quine, but whether Putnam ever believed that the Indispensability Argument established platonism about mathematics remains unclear.[6]

So much then for an introduction to the Indispensability Argument. We have stated the argument informally as well as more formally, and briefly reviewed its origins. Let us now trace some of the philosophical commitments behind the argument's premises before turning to more critical evaluation. The argument indisputably relies on naturalism for its plausibility and, more controversially, on confirmational holism. At any rate, both these principles were championed by Quine, in whose works the argument originated. We examine them in Section 2.

2 Naturalism and Holism

Naturalism is one of those catchwords, like freedom or democracy, that can mean virtually anything to anyone and in which just about everyone professes to believe. Adjectives help discipline the notion. According to *metaphysical* naturalism, the ontology of the world is in some sense "natural." Let us not dwell on what exactly that might mean, since it is another type of naturalism that is most relevant here: *methodological* naturalism. Methodological naturalism enjoins taking the epistemic standards/methods of the natural sciences as primary. As Quine (1981, p. 21) put it, "naturalism: the recognition that it is within science itself, and not in some prior philosophy, that reality is to be identified and described."

Quine never articulated methodological naturalism much more precisely. And he said little more about the scientific method than that it is made up of the norms of empirical adequacy and theoretical simplicity, scope, fertility, and familiarity. So what exactly might be meant by the primacy of scientific methods?

2.1 Methodological Naturalism

Here is a strong form of (methodological) naturalism:

> *Biconditional Naturalism*: One should believe *p* iff science endorses *p*.

By the term of art "science endorses *p*" we mean, roughly, that *p* follows from the tenets of a particular science along with observational statements via an

[6] See the later disavowal in Putnam (2012, pp. 181–3), where he backs away from ontological conclusions and even suggests that he never endorsed them. Putnam (1967) suggests that platonism and a non-platonist picture of mathematics (which later came to be known as "modal structuralism") are mathematically equivalent and equally satisfactory overall (though in some contexts one may be preferable to the other). Liggins (2008) expands on the differences between Quine and Putnam's versions of the Indispensability Argument, as does Putnam (2012) himself in chapter 9. Colyvan (2001, chapter 1) discusses the Indispensability Argument's formulation in more detail.

acceptable process of inference (which might include deduction, induction, or abduction). What exactly counts as a scientific tenet and an acceptable process of inference may not always be entirely clear. Modulo these clarifications, the version of naturalism just stated and others to be canvassed express well-defined norms.

We do not know of any contemporary biconditional naturalists, for good reason.[7] For if science does not speak to a question, or at least does not return an unequivocal answer to it, there may still be sufficiently strong evidence to believe or disbelieve a particular answer to it. Possible examples include whether God exists or what right moral action consists in.

A somewhat more modest version of naturalism is:[8]

> *Trumping Naturalism*: If science endorses *p*, one should believe *p*.

In contrast to the biconditional version, this trumping version has many supporters. For instance, Burgess and Rosen express their naturalism as follows: "The naturalists' commitment is . . . to the comparatively modest proposition that when science speaks with a firm and unified voice, the philosopher is either obliged to accept its conclusions or to offer what are recognizably scientific reasons for resisting them" (1997, p. 65). Many others have upheld Trumping Naturalism, or something in its vicinity.[9]

Trumping Naturalism (or something like it) seems to animate the Indispensability Argument, specifically its first premise. If a collection of claims is part of our best present scientific theories, and omitting these claims from our theories would render the theories scientifically inferior, then we should be committed to the claims in question. There is no vantage point outside science from which to criticize the established findings of science – no "first philosophy prior to natural science," as Quine disparagingly called it.[10] So if best science indispensably uses mathematics, there can be no good reasons from outside science to reject the truth of mathematics.

Trumping Naturalism, however, is too strong a thesis. The history of science counsels that it would be foolhardy to commit ourselves to currently leading

[7] Perhaps Quine was one. Although Mark Colyvan's statement of one of the Indispensability Argument's premises is a version of Biconditional Naturalism, we doubt he espouses anything quite this strong. His formulation of the premise, with italics added to highlight its biconditional nature, is that "We ought to have ontological commitment to *all and only* the entities that are indispensable to our best scientific theories" (Colyvan 2019, §1).

[8] We prefer "Trumping Naturalism" to "Conditional Naturalism" for vividness.

[9] See the many references in Paseau (forthcoming a) or Daly and Liggins (2011).

[10] In, for example, Quine (1981, p. 67).

scientific theories without reservation; it suggests rather that these theories are at best only approximately true. For example, such was the success of Newtonian mechanics in the eighteenth and nineteenth centuries that it was generally deemed not just true, but definitively true. Einstein's 1905 special theory of relativity shattered this confidence. Newtonian mechanics, it is now thought, gives very close though not entirely exact predictions, so in this sense closely approximates the truth in many contexts. Yet its ontology of absolute space and time has been repudiated, so it also contains claims now regarded as outright false. Writers such as Penelope Maddy have also stressed the role of idealizations in science, which are not regarded as literally true (more on this in Section 4.1).

The situation is somewhat different with mathematics. Here the picture of strictly cumulative progress is on safer ground. That much of currently applied mathematics is true is easier to defend. Specifically in connection with the replacement of one scientific theory by another, Bangu (2012, chapter 9) points out that mathematics features in the theories of the workings of all but the most basic observation instruments (telescopes, microscopes, etc.). These instruments are used to collect the data on the basis of which it is argued that a currently held theory should be replaced by a proposed successor. So even if, as some would have it, much science is later shown to be false, the very mechanism by which that is done usually leaves intact the applied mathematics it uses. This is another challenge to the Indispensability Argument that we shall come back to.

What we have called Trumping Naturalism was, for Quine, a fundamental commitment not susceptible to further justification.[11] Others have sought to justify it directly, mostly in terms of track-record considerations. These attempted justifications observe that recent science has been very successful whereas, time and again, philosophy and other nonscientific disciplines have failed. In particular, in cases of conflict, science has a better track record than other forms of inquiry. Many philosophers sympathetic to this argument have cited approvingly David Lewis' credo to the effect that it would be absurd to reject mathematics on philosophical grounds (Lewis 1991, pp. 58–9).[12] Although Lewis' focus was on mathematics, the naturalist's sentiment extends to scientists of any stripe.

11 Though Quine did not call it that, nor did he call his brand of naturalism methodological, and as noted earlier he appeared to embrace the even stronger Biconditional Naturalism.

12 A famous example is the rejection of classical mathematics by the Dutch intuitionist L. E. J. Brouwer. One can interpret him as holding that mathematical objects are mental rather than abstract. Brouwer went on to build a radically novel mathematics on the basis of this philosophical conviction.

Critics of the track-record argument for naturalism allege that it cannot justify a form of naturalism as strong as the trumping version. Daly and Liggins (2011), who call the kind of respect for science at issue here "deferentialism," argue, against Lewis, that many philosophically motivated revisions to science-cum-mathematics would not clash with practice in any important way.[13] Moreover, they urge, track-record considerations prove too much, since they seem to "discredit the reliability of philosophical grounds for believing anything" (2011, p. 328). One of us (Paseau, forthcoming a) has also argued that a consistent naturalist should not be a dogmatist; they may accord greater weight to scientific than nonscientific considerations, but not absolute weight to the former at the expense of the latter. Moreover, as a result of the considerable disagreement in philosophy, there is no single perspective from which philosophy has a poor track record. (For example, if you are a Berkeleyan idealist you will *not* think idealism was first firmly accepted by philosophy but later discredited.)[14]

In addition to these, there is a telling criticism of simple-minded naturalist attempts to settle traditional philosophical debates, such as whether mathematical objects exist. Even if Trumping Naturalism were true, it would not be the philosophical panacea it purports to be, for there would remain difficult questions about what science endorses *all things considered*. Just because linguistics finds it convenient to assume that "Mother Teresa was a good person" is a truth-valued sentence, for example, does not mean that science does so all things considered. Whether we should regard the sentence as truth-valued based on science as a whole remains just as stubborn a question as ever, since we will have to take in much more than narrowly linguistic considerations.[15] Another way to put the point is that, since indispensability is an all-things-considered notion, it cannot be settled by a superficial look at science. The pros and cons of various interpretations of science must be carefully assessed.

To see how this last point plays out in the context of the Indispensability Argument, we note, as we did earlier, that the natural sciences make heavy use of mathematics. They appear to refer to and quantify over numbers, functions, geometric shapes and solids, sets, and the like. However, suppose that questions of mathematical ontology should ultimately be settled by scientific considerations. One might then argue that (i) the fewer types of mathematical objects posited the better, and that (ii) the principle of ontological economy

[13] Daly and Liggins focus on mathematics and linguistics but their point generalizes.

[14] Of course, that raises the issue of why there is more disagreement in philosophy than in science, and what this shows about philosophy's credibility. But this "disagreement" or "lack of convergence" argument is distinct from the track-record argument. For more, see Paseau (forthcoming, a).

[15] Paseau (forthcoming, a) and (2005) press versions of this point.

expressed in (i) is a tenet of scientific theory choice. Or, as mentioned earlier, one might argue that although science endorses the truth of accepted mathematics, it does not endorse the existence of mathematical objects.[16] Or, at any rate, that determining whether it does requires weighing up the pros and cons of various positions, proceeding very much in the manner of contemporary philosophy of mathematics.[17] For example, platonism may not follow even if, on the surface, accepting Maxwell's equations (fundamental to electromagnetism) commits you to abstract differential operators. Broadly scientific considerations may show that the equations' surface reading is not the correct one.

Finally, even if science does endorse an ontology of abstract objects, it may not straightforwardly endorse a particular one.[18] Would physics really be worse off, for example, if mathematical objects turned out to be categories or objects in a category rather than sets? It seems not. Although naturalism gives us a steer on these thorny debates, its mere invocation is not enough to get us out of the briar patch of philosophical(-like) controversy.

To recap, the Indispensability Argument's first premise is supported by Trumping Naturalism. Whether Trumping Naturalism can be motivated by track-record considerations remains unclear, dubious even. And even if Trumping Naturalism is true, there remains an awful lot of philosophical work to do to determine what exactly it is that science endorses.

Still, even if nothing quite as strong as Trumping Naturalism is true, most philosophers today – and that includes us – would want to give a *lot* of weight to scientific considerations. If it turns out that science cannot be done well without assuming abstract objects, that would be a strong reason to believe in abstract objects. Not an indefeasible reason, but a very strong one nonetheless. This slightly weaker version of naturalism than the trumping version supports a claim that falls only a little short of the Indispensability Argument's first premise. It supports not quite the claim that we ought rationally to be ontologically committed to scientifically indispensable objects, but rather that there are very strong rational grounds to be ontologically committed to them.

[16] See Paseau (2007).
[17] For example, weighing up the respective merits of platonism and eliminative structuralism. Roughly, eliminative structuralism takes any mathematical statement as a claim about what holds in any structure satisfying some axioms. For example, arithmetic is not about the natural numbers but about any structure that satisfies the axioms of arithmetic (usually taken to be the Dedekind–Peano axioms). This form of structuralism is eliminative because it does not posit any objects to back up the truth of thus-interpreted mathematical claims.
[18] See chapter 12 of Paseau (forthcoming, b).

2.2 Maddy's Second Philosophy

Penelope Maddy is probably the most influential contemporary naturalist in the philosophy of mathematics, so a quick summary of her views is in order. (This short section may be skipped without loss of continuity.) In a series of publications, most notably in her book *Second Philosophy*, Maddy (2007) has developed a form of naturalism which she calls "Second Philosophy." In a nutshell, the Second Philosopher begins with observation and experimentation and progresses from there to theory formation and testing. Improving and correcting her account of nature by this back-and-forth dialogue between observation and theory, she eventually reaches questions we could classify as philosophical. Examples include whether we can hold reliable beliefs about the external world or whether mathematical objects exist. Second Philosophy consists in the answers such a character would give to those questions. And as Maddy depicts her, the Second Philosopher lacks a principled distinction between "science" and "nonscience"; consequently, she cannot so much as state the trumping naturalist's credo.[19]

Maddy is surely right that the line between science and nonscience is not easy to draw. However, Maddy's Second Philosopher does engage with questions typically classified as philosophical, and when doing so she returns answers more or less identical to those a self-avowed trumping naturalist would. The Second Philosopher thus proceeds piecemeal, behaving much as a trumping naturalist might, but without subscribing to a global naturalist doctrine. The question is whether this refusal to embrace a global expression of her epistemic behavior is a satisfactory stance – at least for a reflective Second Philosopher. We find this question an interesting and important one but here we must put it to one side, as it does not directly affect the rest of the discussion.

2.3 Confirmational Holism

As we saw in Section 1, the Indispensability Argument stems from Quine, who subscribed to naturalism (Section 2.1). Quine also subscribed to another doctrine, which many believe props up the Indispensability Argument: confirmational holism.[20] According to it, the unit of justification is a cluster of theories rather than a single hypothesis – or, in an extreme version, the whole of science.

[19] The summary in this paragraph lightly paraphrases a passage in Maddy (forthcoming).

[20] Also known as justificatory or epistemological holism, and not to be confused with semantic holism (the idea that the meaning of a sentence depends on the meaning of all other sentences in the language). We use the pairs of words – such as "justificatory" and "confirmational," "justify" and "confirm" – interchangeably. An excellent account of Quine's philosophy of mathematics may be found in Resnik (2005).

Confirmational holism is sometimes known as the Duhem–Quine thesis, after Quine and Pierre Duhem.[21] The Duhem–Quine thesis has its source in the insight that scientific statements imply observational claims only in conjunction with auxiliary hypotheses. Let H and O respectively denote a hypothesis and an observation statement, and let each A_i (with index i) be an auxiliary statement. To say that observation statements entail hypotheses only in conjunction with auxiliary statements is to say that $H \,\&\, A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_n$ entail O but that H does not do so on its own. The Duhem–Quine thesis takes this apparent fact about entailment (or scientific prediction) and flips it into a thesis about justification. An observational statement such as O justifies not H on its own but the conjunction $H \,\&\, A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_n$. Conversely, a contrary observation does not justify the rejection of a single hypothesis H; rather, it justifies the rejection of $H \,\&\, A_1 \,\&\, A_2 \,\&\, \ldots \,\&\, A_n$, that is, at least one of $H, A_1, A_2, \ldots, A_n$. As Quine puts it:

> Suppose an experiment has yielded a result contrary to a theory currently held in some natural science. The theory comprises a whole bundle of conjoint hypotheses, or is resoluble into such a bundle. The most that the experiment shows is that at least one of these hypotheses is false; it does not show which. It is only the theory as a whole, and not any of the hypotheses, that admits of evidence or counter-evidence in observation and experiment. (Quine 1970, p. 5)

Take, for example, Sir Arthur Eddington's eclipse experiment in May 1919, designed to test which (if any) of Newtonian mechanics or Einsteinian general relativity is correct. Einstein's theory predicted that at the moment of the eclipse, light rays from stars would be deflected by twice the amount predicted by Newton's theory. There were two observation stations, one in Brazil and one in Príncipe (off the west coast of Africa). Photographs from Príncipe were dim but could, on the back of some complex calculations, be interpreted as favoring Einstein's theory. Photographs from one of the Brazilian telescopes suggested an Einsteinian shift, but photographs from the second Brazilian telescope indicated a Newtonian one. To further complicate matters, the sun's heating systematically biased both Brazilian telescopes, or so Eddington argued. In popular accounts, Eddington's expedition is often presented as a crucial experiment to cleanly test the relative merits of Newton and Einstein's theories of gravitation. But clearly it did no such thing: at best, it tested those theories combined with a host of auxiliary assumptions about telescopes' optical properties, their thermal properties, the positions of the stars, and so on.[22]

[21] For Duhem, see in particular his 1906/2007 work.
[22] We have drawn on the fascinating account of Eddington's experiment in chapter 2 of Strevens (2020).