Elliptic Partial Differential Equations

This chapter is an introduction to elliptic partial differential equations (PDEs). We start with elementary notions and basic results. Then we proceed with a discussion on foundational regularity results for this class of equations.

1.1 Basic Definitions and Facts

We are concerned with second-order equations of elliptic type, whose general formulation is

\[ G(x, u, Du, D^2u) = 0. \]

Here, \( G = G(x, r, p, M) \) stands for a rule relating the Hessian of the unknown \( D^2u \), its gradient \( Du \), the unknown \( u \) itself and the spatial variable \( x \). In the following, we examine examples of elliptic equations.

**Example 1.1** (Linear equations in nondivergence form) Let \( \Omega \subset \mathbb{R}^d \) be an open and connected subset of \( \mathbb{R}^d \). Denote by \( S(d) \) the space of \( d \times d \) symmetric matrices. A linear equation in nondivergence form can be written as

\[-\text{Tr}(A(x)D^2u) + b(x) \cdot Du + c(x)u = 0 \quad \text{in} \quad \Omega, \quad (1.1)\]

where \( A: \Omega \to S(d) \) is a matrix-valued map, \( b: \Omega \to \mathbb{R}^d \) is a vector field, and \( c: \Omega \to \mathbb{R} \) is a scalar function. In this concrete case, \( G \) takes the form

\[ G(x, r, p, M) = -\text{Tr}(A(x)M) + b(x) \cdot p + c(x)r. \]

If there are constants \( 0 < \lambda_0 \leq \lambda \leq \Lambda \) such that \( A(x) \) satisfies

\[ \lambda |\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda |\xi|^2, \]

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for every $\xi \in \mathbb{R}^d$, we say that $A(\cdot)$ is a $(\lambda, \Lambda)$-elliptic matrix. In this case, (1.1) is a uniformly elliptic equation. If instead it holds $0 \leq \lambda$, we say that (1.1) is merely elliptic. To make the distinction more apparent, suppose $d = 2$, make $b \equiv 0$ and $c \equiv 0$, and let $A(x)$ be given by

$$A(x) := \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}.$$ 

The equation becomes

$$x_1 \frac{\partial^2 u}{\partial x_1 \partial x_1} + x_2 \frac{\partial^2 u}{\partial x_2 \partial x_2} = 0;$$

if we prescribe this problem in the first quadrant $\mathbb{R}^2_+$ it is elliptic, although not uniformly elliptic. However, if the equation is supposed to hold in a subset $\Omega \subset \mathbb{R}^2_+$ strictly away from the axis (i.e., with distance strictly greater than some $\lambda > 0$), the problem becomes uniformly elliptic.

In either case, the operator in (1.1) satisfies a monotonicity property with respect to the matrix $M$. Indeed, fix $p \in \mathbb{R}^d$ and $r \in \mathbb{R}$ and let $M, N \in S(d)$ be such that $N - M \geq 0$. That is,

$$(N - M)\xi \cdot \xi \geq 0,$$

for every $\xi \in \mathbb{R}^d$. Then

$$G(x, r, p, M) - G(x, r, p, N) = \operatorname{Tr} (A(x)(N - M)) \geq 0. \quad (1.2)$$

If we set $b \equiv 0$ and $c \equiv 0$ in (1.1), we recover a purely second-order equation; if we further restrict the operator by requiring $A(x)$ to be the identity matrix $I$ we get the Laplace operator.

**Example 1.2** (Linear equations in divergence form) A linear equation in the divergence form can be written as

$$-\frac{\partial}{\partial x_j} \left( a^{ij}(x) \frac{\partial u}{\partial x_i} \right) + \frac{\partial}{\partial x_j} (b^j(x)u) + c(x)u = 0 \quad \text{in} \quad \Omega, \quad (1.3)$$

where $A(x) := (a^{ij}(x))_{i, j = 1, \ldots, d}$ is a matrix-valued map, $b(x) = (b^1(x), \ldots, b^d(x))$ is a vector field, and $c(x)$ is a scalar function; $A(\cdot), b(\cdot),$ and $c(\cdot)$ are defined over the domain $\Omega$. An alternative way to write the operator in (1.3) is

$$-\operatorname{div}(A(x)Du) + \operatorname{div}(b(x)u) + c(x)u = 0.$$ 

As before, the ellipticity of the problem is governed by the behavior of the matrix $A$. The equation is elliptic if $A$ is nonnegative, whereas it is uniform elliptic if
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\[ \lambda |\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda_1 |\xi|^2, \]
for every \( \xi \in \mathbb{R}^d \). If the coefficients \( A(\cdot), b(\cdot), \) and \( c(\cdot) \) are regular enough — say, of class \( C^2 \) — it is possible to write (1.3) in the form of (1.1). In fact, in the case where we are allowed to compute the derivatives in (1.3), it becomes

\[-\text{Tr}(A(x)D^2u) + \tilde{b}(x) \cdot Du + \tilde{c}(x)u = 0 \quad \text{in} \quad \Omega,\]

where

\[ \tilde{b}^i := b^i + (a_{ij})x_j \quad \text{and} \quad \tilde{c}(x) := c(x) + b^j, \]

By setting \( A(x) \equiv I, b \equiv 0, \) and \( c \equiv 0, \) once again we recover the Laplace operator

\[ -\text{div}(Du) = -\Delta u. \]

The analysis of Examples 1.1 and 1.2 suggests, we can perturb the Laplace operator in (at least) two ways. Set \( b \equiv 0 \) and \( c \equiv 0. \) Let \( \varphi \in C^\infty_c(\Omega, \mathbb{R}^d) \) and consider

\[ A_\varepsilon(x) := \begin{bmatrix}
1 + \varepsilon \varphi_1(x) & 0 & \ldots & 0 \\
0 & 1 + \varepsilon \varphi_2(x) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 + \varepsilon \varphi_d(x)
\end{bmatrix}, \]

for \( 0 < \varepsilon \ll 1, \) sufficiently small. For every \( \varepsilon > 0 \) one can detach from the Laplace operator either in a nondivergent form, by taking

\[ L[u] := -\text{Tr}(A_\varepsilon(x)D^2u), \]

or in a divergent form, by considering

\[ L[u] := -\text{div}(A_\varepsilon(x)Du). \]

A question arising in that exercise concerns qualitative properties one can transmit from the Laplace operator to the perturbed models governed by \( A_\varepsilon. \) A more subtle issue concerns the amount of information one can import, depending on the form of the perturbation (nondivergence or divergence form). We return to those questions further in the book.

In addition to the notion of classical solution — a function \( u \in C^2(\Omega) \) satisfying an equation in the classical sense — we consider weak solutions to the PDEs under analysis. In the context of operators in the divergence form, we use the notion of weak solutions in the distributional sense.
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**Definition 1.3** (Weak distributional solutions) Let $A \in L^2_{\text{loc}}(\Omega, S(d))$, $b \in L^2_{\text{loc}}(\Omega, \mathbb{R}^d)$, $c \in L^2_{\text{loc}}(\Omega)$, and $f \in L^1_{\text{loc}}(\Omega)$. We say that $u \in W^{1,2}_{\text{loc}}(\Omega)$ is a weak solution to

$$
- \text{div}(A(x)Du) + \text{div}(b(x)u) + c(x)u = f \quad \text{in} \quad \Omega,
$$

if, for every $\varphi \in C_\infty^c(\Omega)$, we have

$$
\int_\Omega (A(x)Du - b(x)u) \cdot D\varphi + \int_\Omega c(x)u\varphi = \int_\Omega f\varphi.
$$

We refer to (1.5) as the weak form of (1.4). The test function $\varphi$ used in Definition 1.3 can be taken in different functional spaces when appropriate. To write the weak form (1.5) we only require the coefficients of the equation to be in appropriate $L^2$-spaces.

In the context of divergence form problems, it is sometimes convenient to write equations as

$$
- \text{div} a(x, u, Du) = 0 \quad \text{in} \quad \Omega,
$$

where $a: \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is a suitable vector field. Here, the notion of weak distributional solution adjusts in the obvious way. For instance, suppose

$$
|a(x, r, p)| \leq C \left( 1 + |r|^q + |p|^{r'} \right),
$$

for some $C > 0$ and $1 < q \leq r$; hence, $u \in W^{1,r}_{\text{loc}}(\Omega)$ is a weak solution to (1.6) if

$$
\int_\Omega a(x, u, Du) \cdot D\varphi = 0,
$$

for every $\varphi \in C_\infty^\infty(\Omega)$. The formulation in (1.6) allows us to consider diffusion coefficients depending also on $u$ and $Du$, even in a nonlinear way. This observation leads us to our next example.

**Example 1.4** (The $p$-Laplace operator) Let $p > 1$ be fixed and consider

$$
\text{div} \left( |Du|^{p-2}Du \right) = f \quad \text{in} \quad \Omega,
$$

for some function $f \in L^q(\Omega)$, with $q \in (1, \infty]$. We denote the operator in (1.7) by $\Delta_p$ and write the associated equation as $\Delta_p u = f$. The $p$-Laplace describes diffusion processes whose coefficients depend on the gradient of the solutions. At the points where $Du$ vanishes, the ellipticity of the equation collapses. If $p > 2$, the problem degenerates, whereas in the case $p < 2$ it becomes singular.
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Clearly, the case $p = 2$ recovers the Laplace operator. We observe that (1.7) is the Euler–Lagrange equation associated with the $L^p$-energy functional

$$I_p[u] := \int_\Omega \frac{|Du|^p}{p} + uf \, dx.$$ 

In analogy with the case $p = 2$, we sometimes refer to the solutions to $\Delta_p u = 0$ as $p$-harmonic functions.

Setting $p = 1$ in (1.7) we discover an intrinsically geometric equation. In fact, we get

$$\text{div} \left( \frac{1}{|Du|} Du \right) = 0$$

in $\Omega$. For every $\alpha \in \mathbb{R}$, suppose the level set \{ $u = \alpha$ \} is a smooth hypersurface; hence, its mean curvature is given by the divergence of the normal unit vector. The conclusion is that the level sets of $u$ have zero mean curvature; i.e.,

$$\Gamma_\alpha(u) := \{ u = \alpha \}$$

is a minimal surface for every $\alpha \in \mathbb{R}$.

Another instance of interest is the case $p \to \infty$; we examine it here from a heuristic viewpoint. Suppose $u \in C^2(\Omega)$ is $p$-harmonic and compute

$$0 = \text{div} \left( |Du|^{p-2} Du \right) = |Du|^{p-2} \Delta u + (p - 2)|Du|^{p-4} u_{x_i} u_{x_j} u_{x_i x_j};$$

dividing both sides of the former equality by $|Du|^{p-2}(p - 2)$ we obtain

$$\frac{\Delta u}{p-2} + \frac{1}{|Du|^2} u_{x_i} u_{x_j} u_{x_i x_j} = 0.$$ 

As $p \to \infty$ the operator becomes

$$\Delta_\infty u = D^2 u Du \cdot Du = 0.$$ 

Similar to what happens in the case $1 < p < \infty$, the $\infty$-Laplacian can also be regarded as an Euler–Lagrange equation. As one could expect, it relates to an $L^\infty$-type of energy. Indeed, let $w : \partial \Omega \to \mathbb{R}$ be a Lipschitz continuous function. We seek $u : \Omega \to \mathbb{R}$ coinciding with $w$ on $\partial \Omega$ and such that

$$\text{ess sup}_{x \in \Omega'} |Du(x)| \leq \text{ess sup}_{x \in \Omega'} |Dw(x)|,$$

for every open set $\Omega' \subset \Omega$.

We note the notion of weak distributional solutions is appropriate for the $p$-Laplace operator. However, it does not seem useful when considering the $\infty$-Laplace. In general, when turning our attention to nondivergent equations, as in Example 1.1, the notion of a weak distributional solution is not adequate. In this context we resort to the concept of the viscosity solution.
For ease of presentation, consider an operator \( F : \Omega \times \mathbb{R} \times \mathbb{R}^d \times S(d) \to \mathbb{R} \). Suppose that for \((x, r, p)\) arbitrary and \(M, N \in S(d)\), with \(M - N \geq 0\), we have
\[
F(x, p, r, M) \leq F(x, p, r, N). \tag{1.8}
\]
An operator satisfying (1.8) is called degenerate elliptic.

**Definition 1.5** (*C*-viscosity solutions) Suppose \( F \in C(\Omega \times \mathbb{R} \times \mathbb{R}^d \times S(d)) \). A function \( u \in USC(\Omega) \) is a \( C \)-viscosity subsolution to
\[
F(x, u, Du, D^2 u) = 0 \quad \text{in} \quad \Omega, \tag{1.9}
\]
if, for every \( \varphi \in C^2(\Omega) \) and \( x_0 \in \Omega \) such that \((u - \varphi)(x_0) \geq (u - \varphi)(x)\) locally, we have
\[
F(x_0, u(x_0), D\varphi(x_0), D^2 \varphi(x_0)) \leq 0.
\]
Conversely, we say that \( u \in LSC(\Omega) \) is a \( C \)-viscosity supersolution to (1.9) if, for every \( \varphi \in C^2(\Omega) \) and \( x_0 \in \Omega \) such that \((u - \varphi)(x_0) \leq (u - \varphi)(x)\) locally, we have
\[
F(x_0, u(x_0), D\varphi(x_0), D^2 \varphi(x_0)) \geq 0.
\]
Finally, if \( u \in C(\Omega) \) is both a \( C \)-viscosity subsolution and a supersolution to (1.9), we say it is a \( C \)-viscosity solution to (1.9).

The notion of viscosity solutions recasts important properties of classical solutions in the context where \( C^2 \)-regularity is not available. For example, suppose \( F = F(M) \) is a degenerate elliptic operator and \( v \in C^2(\Omega) \) is a classical solution to
\[
F(D^2 v) = 0 \quad \text{in} \quad \Omega.
\]
Take \( \varphi \in C^2(\Omega) \) and \( x_0 \in \Omega \) such that \( v - \varphi \) attains a maximum at \( x_0 \). Clearly \( D^2 v(x_0) - D^2 \varphi(x_0) \leq 0 \). Hence, \( F(D^2 \varphi(x_0)) \leq F(D^2 v(x_0)) = 0 \), and the test function \( \varphi \) satisfies the inequality characterizing \( C \)-viscosity subsolutions. We will discuss further properties of \( C \)-viscosity solutions throughout the book. We proceed with an example.

**Example 1.6** (Pucci extremal operators) Let \( 0 < \lambda \leq \Lambda \) be fixed constants. Define the operators \( \mathcal{P}^{\pm}_{\lambda, \Lambda} : S(d) \to \mathbb{R} \) as
\[
\mathcal{P}^{\pm}_{\lambda, \Lambda}(M) := -\lambda \, \text{Tr} \left(M^+\right) + \Lambda \, \text{Tr} \left(M^-\right)
\]
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and

\[ P_{-\lambda, \Lambda}(M) := -\Lambda \text{Tr} (M^+) + \lambda \text{Tr} (M^-), \]

where \( M^+ \) is the positive semidefinite part of the matrix \( M \) and \( M^- \) is its negative semidefinite part. Although \( P_{-\lambda, \Lambda}^\pm \) are nonlinear, a simple calculation shows these operators are positively homogeneous of degree one. As a consequence, we learn that \( P_{-\lambda, \Lambda}^\pm \) can not be differentiable operators. Indeed, were \( P_{-\lambda, \Lambda}^\pm \) differentiable, their gradients would be homogeneous of degree zero and, therefore, constant maps. From this fact, we would infer the extremal operators are linear. By setting \( \lambda = \Lambda = 1 \), we have \( P_{-\lambda, \Lambda}^\pm(M) = \text{Tr} M \) and recover the Laplace operator.

Pucci operators play a central role in the analysis of nonlinear elliptic equations. They are used to characterize classes of viscosity solutions and in the statement of results aiming at holding for general, abstract operators. We make extensive use of extremal operators in the next chapters. We continue with yet another example.

**Example 1.7** (Fully nonlinear elliptic operator) Let \( N \subset \Omega \) be a null set, with respect to the Lebesgue measure. Let \( F : (\Omega \setminus N) \times \mathbb{R} \times \mathbb{R}^d \times S(d) \to \mathbb{R} \) be a measurable function. For constants \( 0 < \lambda \leq \Lambda \), suppose the operator \( F = F(x, r, p, M) \) satisfies

\[ F(x, r, p, M) - \Lambda \text{Tr}(P) \leq F(x, r, p, M + P) \leq F(x, r, p, M) - \lambda \text{Tr}(P), \]

(1.10)

for every \((x, r, p) \in (\Omega \setminus N) \times \mathbb{R} \times \mathbb{R}^d \) and \( M, P \in S(d) \), with \( P \geq 0 \). Roughly speaking, (1.10) says that moving from a symmetric matrix \( M \) in the positive direction \( P \), the operator changes proportionally to \( P \) itself. Such changes are controlled by the constants \( \lambda \) and \( \Lambda \). Condition (1.10) is called uniform ellipticity. An operator \( F \) satisfying both inequalities in (1.10) is said to be \((\lambda, \Lambda)\)-elliptic, or uniformly elliptic.

The class of operators introduced in Example 1.7 includes a number of important examples. For instance, the model in Example 1.1 satisfies (1.10), as

\[ -\text{Tr}(A(x)(M + P)) + \text{Tr}(A(x)M) = -\text{Tr}(A(x)P) \geq -\Lambda \text{Tr}(P), \]

provided \( A \) is a \((\lambda, \Lambda)\)-elliptic matrix and \( M, P \in S(d) \), with \( P \geq 0 \). The remainder inequality follows similarly. The Pucci extremal operators also satisfy (1.10); notice that for any two symmetric matrices \( M \) and \( P \), we have \( M + P = M^+ + P^+ \) and \( (M + P)^- = M^- \).
As a result,
\[ P + \lambda, \Lambda_1 (M + P) = -\lambda \text{Tr} [(M + P)^+] + \Lambda \text{Tr} [(M + P)^-] \]
\[ = -\lambda \text{Tr} (M^+ + P^+) + \Lambda \text{Tr} (M^-) \]
\[ = -\lambda \text{Tr} (M^+) + \Lambda \text{Tr} (M^-) - \lambda \text{Tr} (P^+) \]
\[ = P_{\lambda, \Lambda}^+(M) - \lambda \text{Tr}(P). \]

Also
\[ P_{\lambda, \Lambda}^+(M) - \Lambda \text{Tr}(P) = -\lambda \text{Tr} (M^+ + P^+ - P^+) + \Lambda \text{Tr} (M^- + P^- - P^-) - \Lambda \text{Tr}(P^+) \]
\[ = P_{\lambda, \Lambda}^+(M + P) + (\lambda - \Lambda) \text{Tr}(P) \]
\[ \leq P_{\lambda, \Lambda}^+(M + P). \]

We notice that (1.10) implies degenerate ellipticity. In fact, if \( F = F(x, r, p, M) \) satisfies (1.10) and \( M - N \geq 0 \), it follows that
\[ F(x, r, p, M) = F(x, r, p, N + M - N) \leq F(x, r, p, N) - \lambda \text{Tr}(M - N) \leq F(x, r, p, N). \]

An equivalent characterization of uniform ellipticity is possible in terms of matrices’ norms. We say the operator \( F: S(d) \to \mathbb{R} \) is \((\lambda, \Lambda)-elliptic \) if
\[ \lambda \|N\| \leq F(M) - F(M + N) \leq \Lambda \|N\|, \]
for every \( M, N \in S(d) \), with \( N \geq 0 \). The following inequality will be useful further.

**Lemma 1.8** Let \( M, N \in S(d) \) and suppose \( F: S(d) \to \mathbb{R} \) is a \((\lambda, \Lambda)-elliptic \) operator. Then,
\[ F(M) \leq F(M + N) + \Lambda \|N^+\| - \lambda \|N^-\|, \]
for every \( M, N \in S(d) \).

**Proof** For every \( M, N = N^+ - N^- \) we have
\[ F(M + N) \geq F(M + N^+) + \lambda \|N^-\|. \]
Moreover,
\[ F(M + N^+) \geq F(M) - \Lambda \|N^+\|. \]

By combining both inequalities the result follows. \( \square \)

As a consequence of Lemma 1.8 we derive an upper bound for the difference of any two matrices in \( F^{-1}(\{0\}) \).
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**Corollary 1.9**  Let $M, N \in S(d)$ be such that $F(M) = F(N) = 0$. Then

$$\|M - N\| \leq \frac{\Lambda + \lambda}{\lambda} \|(M - N)^+\| = \frac{\Lambda + \lambda}{\lambda} \sup_{\theta \in S^{d-1}} (\theta^T (M - N) \theta)^+.$$

**Proof**  It follows from Lemma 1.8 that

$$0 = F(M) = F(M - N + N) \leq F(N) + \Lambda \|(M - N)^+\| - \lambda \|(M - N)^-\|;$$

i.e.,

$$\lambda \|(M - N)^-\| \leq \Lambda \|(M - N)^+\|.$$

By adding $\lambda \|(M - N)^+\|$ on both sides of the former inequality and applying the triangle inequality we get

$$\lambda \|M - N\| \leq (\Lambda + \lambda) \|(M - N)^\| = (\Lambda + \lambda) \sup_{\theta \in S^{d-1}} (\theta^T (M - N) \theta)^+.$$

□

An additional monotonicity condition involves the dependence of $F$ on the zeroth-order term. From now on, we denote by $N$ the null subset of $\Omega$ comprising the points where $F(\cdot, r, p, M)$ is not defined.

**Definition 1.10** (Properness)  Let $F: (\Omega \setminus N) \times \mathbb{R} \times \mathbb{R}^d \times S(d) \to \mathbb{R}$ be a measurable function. We say that $F(\cdot, r, p, M)$ is proper if it is degenerate elliptic and for every $(x, p, M) \in (\Omega \setminus N) \times \mathbb{R}^d \times S(d)$ we have

$$F(x, r, p, M) \leq F(x, s, p, M),$$

whenever $r \leq s$.

The operator in Example 1.7 is required to be merely measurable, with no further assumptions (e.g., on its continuity). More subtle is the fact that $F(\cdot, r, p, M)$ is defined over a set of full measure; in particular, there might exist points $x_0 \in \Omega$ where the operator is not well-defined. This scenario challenges the notion of $C$-viscosity solutions, as it might not be possible to test the inequalities involved in the definition at some particular point $x_0 \in \Omega$ – simply because the operator might not be defined at that point. To circumvent these difficulties – and also to address further issues in the realm of viscosity solutions – we resort to the notion of the $L^p$-viscosity solution, introduced by Caffarelli et al. (1996).

The definition of the $L^p$-viscosity solution must be consistent with the one for $C$-viscosity solutions, in the sense that if the operator happens to be continuous, they should coincide. On the other hand, the former has to account
for the fact that, a priori, the operator is defined almost everywhere in the domain \( \Omega \). The precise definition reads as follows.

**Definition 1.11** \((L^p\text{-viscosity solution})\) Let \( F : (\Omega \setminus N) \times \mathbb{R} \times \mathbb{R}^d \times S(d) \to \mathbb{R} \) be a measurable function. Suppose \( F \) is proper and let \( f \in L^p(\Omega) \), for \( p > d/2 \).

A function \( u \in \text{USC}(\Omega) \) is an \( L^p \)-viscosity subsolution to

\[
F(x, u, Du, D^2u) = f \quad \text{in} \quad \Omega,
\]

if, for every \( \varphi \in W^{2,p}_\text{loc}(\Omega) \) such that there is \( \varepsilon > 0 \), and an open set \( U \subset \Omega \) for which

\[
F(x, \varphi(x), D\varphi(x), D^2\varphi(x)) - f(x) \geq \varepsilon
\]
a.e.-\( x \in U \), then \( u - \varphi \) does not attain a local maximum in \( U \). A function \( u \in \text{LSC}(\Omega) \) is an \( L^p \)-viscosity supersolution to \((1.11)\) if for every \( \varphi \in W^{2,p}_\text{loc}(\Omega) \) such that there is \( \varepsilon > 0 \), and an open set \( U \subset \Omega \) for which

\[
F(x, \varphi(x), D\varphi(x), D^2\varphi(x)) - f(x) \leq -\varepsilon
\]
a.e.-\( x \in U \), then \( u - \varphi \) does not attain a local minimum in \( U \). If \( u \in C(\Omega) \) is simultaneously an \( L^p \)-viscosity subsolution and a supersolution to \((1.11)\), we say \( u \) is an \( L^p \)-viscosity solution to \((1.11)\).

When dealing with the notion of \( L^p \)-viscosity solutions we require the operator \( F \) to satisfy a structure condition. To make matters precise, we state it in the form of a definition.

**Definition 1.12** (Structure condition) Let \( F : (\Omega \setminus N) \times \mathbb{R} \times \mathbb{R}^d \times S(d) \to \mathbb{R} \) be a measurable function such that \( F(x, \cdot, \cdot, \cdot) \in L^p(\Omega) \), for some \( p > d/2 \).

If there are constants \( 0 < \lambda \leq \Lambda \) and \( \gamma > 0 \), and a modulus of continuity \( \omega_F : \mathbb{R}_+ \to \mathbb{R}_+ \) such that

\[
\mathcal{P}_{\lambda, \Lambda}^-(M - N) - \gamma |p - q| - \omega_F((s - r)^+) \leq F(x, r, p, M) - F(x, s, q, N) \leq \mathcal{P}_{\lambda, \Lambda}^+(M - N) + \gamma |p - q| - \omega_F((r - s)^+),
\]

we say \( F \) satisfies a \((\lambda, \Lambda, \gamma, \omega_F)\)-structure condition.

We notice the structure condition in Definition 1.12 enforces the uniform ellipticity of \( F \). Indeed, take \( M, P \in S(d) \) with \( P \geq 0 \). It follows from the structure condition that

\[
F(x, r, p, M + P) - F(x, r, p, M) \leq \mathcal{P}_{\lambda, \Lambda}^+(P) = -\lambda \operatorname{Tr}(P^+),
\]