Introduction to Proofs and Proof Strategies

Emphasizing the creative nature of mathematics, this conversational textbook guides students through the process of discovering a proof. The material revolves around possible strategies to approaching a problem without classifying "types of proof" or providing proof templates. Instead, it helps students develop the thinking skills needed to tackle mathematics when there is no clear algorithm or recipe to follow. Beginning by discussing familiar and fundamental topics from a more theoretical perspective, the book moves on to inequalities, induction, relations, cardinality and elementary number theory. The final supplementary chapters allow students to apply these strategies to the topics they will learn in future courses. With its focus on "doing mathematics" through 200 worked examples, over 370 problems, illustrations, discussions, and minimal prerequisites, this book will be indispensable to first- and second-year students in mathematics, statistics and computer science. Instructor resources include solutions to select problems.

Shay Fuchs is Associate Professor (Teaching Stream) in the Department of Mathematical and Computational Sciences at the University of Toronto, Mississauga, Canada, and a member of the Mathematical Association of America. He has been a mathematics educator for more than 25 years. His course based on this text has been taken by more than 1500 students and used by dozens of his colleagues in the past 3 years.

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Introduction to Proofs and Proof Strategies

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"Every student in the sciences should be exposed to the basic language of modern mathematics, and standard courses such as calculus or linear algebra do not play this role. The ideal textbook for such a course should not attempt to be encyclopedic and should not assume special prerequisites. It should cover a carefully chosen selection of topics efficiently, engagingly, thoroughly, without being overbearing. Fuchs' text fits this description admirably. The level is right, the math is rock solid, the writing is very pleasant. The book talks to the reader, without ever sounding patronizing. A vast selection of problems, many including solutions, will be splendidly helpful both in a classroom setting and for self-study."

Paolo Aluffi, *Florida State University*

"This well-written text strikes a good balance between conciseness and clarity. Students are led from looking more deeply into familiar topics, such as the quadratic formula, to an understanding of the nature, structure, and methods of proof. The examples and problems are a strong point. I look forward to teaching from it."

Eric Gottlieb, *Rhodes College*

"Fuchs' text is an excellent addition to the 'transitions to proof' literature. I will use it when I next teach such a course. Except for the excellent 'Additional Topics' sections, the content is standard, but the spiraling presentation and helpful narrative around proofs are what truly elevate this text. Fuchs has made every attempt to connect the structure and rigor of mathematics with the intuition of the student. For example, the notion of function arises in three different chapters, with two increasingly rigorous 'provisional definitions,' before a complete definition is given within a wider discussion of relations. I anticipate this approach resonating with students. Fuchs' Chapter 3, which introduces logic and proof strategies, is the most usable presentation of the material I have seen or used. The practice of mathematics and mathematical thinking is communicated well, while opportunities for confusion and obfuscation via a blizzard of symbols are minimized."

Ryan Grady, *Montana State University*

"This book is a must-have resource for an undergraduate mathematics student or interested reader to learn the fundamental topics in how to prove things. The text is thorough and of top quality, yet it is conversational and easy to absorb. Maybe the most important quality, it offers advice about how to approach problems, making it perfect for an introduction to proofs class."

Andrew McEachern, *York University, Canada*

"This is a great choice of textbook for any course introducing undergraduates to mathematical proofs. What makes this book stand out are the early chapters, as well as the 'Additional Topics,' both with accompanying exercises. The book begins by gently introducing proof-based thinking by posing well-motivated prompts and exercises concerning familiar arithmetic of real numbers and the integers. It then introduces fields as a playground to practice working with axioms and drawing (sometimes surprising) conclusions from them. The book proceeds with introducing formal logic, mathematical induction, set theory, and relations on sets. The book's design nicely enables framing classes around a choice sampling among the abundant exercises. The book's 'Additional Topics' can serve to engage those students with a brimming imagination and who are already familiar with basic notions of proofs."

David Ayala, *Montana State University*

"Fuchs' *Introduction to Proofs and Proof Strategies* is an excellent textbook choice for an undergraduate proof-writing course. The author takes a friendly and conversational approach, giving many worked examples throughout each section. Furthermore, each section is replete with exercises for the reader, along with fully worked solutions at chapter's end. This is exactly the 'get your hands dirty' approach students and readers will benefit greatly from!"

Frank Patane, *Samford University*

"The book *Introduction to Proofs and Proof Strategies* by Shay Fuchs takes the problem-solving approach to the forefront by accompanying the reader in the construction and deconstruction of proofs through numerous examples and challenging exercises. The fundamental principles of mathematics are introduced in a creative and innovative way, making learning an enjoyable journey."

Roberto Bruni, *Università di Pisa*

"This textbook is easy to read and designed to enhance students' problem-solving skills in their first year of university. The book really stands out due to the variety and quality of exercises at the end of each chapter. The latter chapters dive into more advanced topics for interested students."

Marina Tvalavadze, *University of Toronto Mississauga*

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SYMBOLS AND NOTATION

Δ	discriminant of a quadratic equation
$\square$	end of proof
$\mathbb{C}$	the field of complex numbers
$\mathbb{N}$	the set of natural numbers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{Z}$	the set of integers
$\lvert x\rvert$	the absolute value of x
$[a,b]$	a closed interval
(a,b)	an open interval
∞ or $+\infty$	positive infinity
$-\infty$	negative infinity
$\sum$	summation (Sigma) notation
$\prod$	product (Pi) notation
$n!$	n factorial
$b \mid a$	b divides a (or a is divisible by b)
$b \nmid a$	b does *not* divide a
$\gcd(a,b)$	the greatest common divisor of a and b
$\operatorname{lcm}(a,b)$	the least common multiple of a and b
$[x]$	the equivalence class of x
$a \equiv b \pmod{n}$	a and b are congruent modulo n
$\binom{n}{k}$	the binomial coefficient n *choose* k
$\lim a_n$	the limit of the sequence (a_n)
$N_r(c)$	an r-neighbourhood of c
$N_r^*(c)$	a punctured r-neighbourhood of c
$\{\cdots\}$	a set
$y \in C$	y is an element of the set C
$y \notin C$	y is *not* an element of the set C
$C \subseteq D$ or $D \supseteq C$	C is a subset of D

$C \not\subseteq D$ or $D \not\supseteq C$	C is a *not* subset of D
$\emptyset$	the empty set
$A \cap B$	the intersection of A and B
$A \cup B$	the union of A and B
$A \setminus B$	the difference between A and B
A^c	the complement of the set A
$A \times B$	the Cartesian product of A and B
$S/\sim$	the quotient set (or space)
$\bigcap_{\alpha \in J} A_\alpha$	the intersection of sets A_α for $\alpha \in J$
$\bigcup_{\alpha \in J} A_\alpha$	the union of sets A_α for $\alpha \in J$
$P(X)$	the power set of X
$\lvert A \rvert$	the number of elements in the finite set A
$\lvert A \rvert = \lvert B \rvert$	the sets A and B have the same cardinality
$\lvert A \rvert \leq \lvert B \rvert$	A has cardinality less than or equal to the cardinality of B
$\lvert A \rvert \geq \lvert B \rvert$	A has cardinality greater than or equal to the cardinality of B
$\forall$	for all/for every (the universal quantifier)
$\exists$	there is/there exists (the existential quantifier)
$\neg$	not (negation)
$\wedge$	and (conjunction)
$\vee$	or (disjunction)
$\Rightarrow$	if-then (implication)
$\Leftrightarrow$	if-and-only-if (equivalence)
$f : A \to B$	f is a function with domain A and codomain B
$f(C)$	the image of the set C under the function f
f^{-1}	the inverse function of f
$f^{-1}(D)$	the pre-image of the set D under the function f
$\lim_{x \to c} f(x)$	the limit of the function f as x approaches c
$\lim_{x \to c^-} f(x), \lim_{x \to c^+} f(x)$	the left- and right-hand limits of f as x approaches c
$f'(c)$	the derivative of the function f at c
$Ker(T), Im(T)$	the kernel and image of a linear map T
i	the complex imaginary unit satisfying $i^2 = -1$
$Re(z), Im(z)$	the real and imaginary parts of the complex number z
$\bar{z}$	the conjugate of the complex number z
$\underline{u}, \underline{v}, \underline{w}$	vectors in $\mathbb{R}^n$
$\overrightarrow{AB}$	a geometric vector from A to B
$\lvert\lvert \underline{v} \rvert\rvert$	the length of a vector in $\mathbb{R}^n$

PREFACE

The main purpose of this book is to help students develop problem-solving and mathematical thinking skills required for the advanced study of mathematics and related fields. Quite often, high school mathematics focuses on applying techniques, carrying out computations and following prescribed algorithms, which are insufficient for being successful in post-secondary mathematics and mathematics-related disciplines. This book presents mathematics as a creative field, which involves experimentation and the development of new ideas as part of the journey.

In terms of content, the first seven chapters cover fundamental topics in mathematics, such as sets, functions, relations and more. The remaining four chapters cover additional topics that can be introduced as a preview for upcoming courses or as a place for formalizing ideas presented more intuitively in previous courses. Chapters 8–11 are independent of each other, and so the instructor may choose to cover any of them, in any order, or none of them. Our recommendation is that a one-semester course cover Chapters 1–7, and then, as time permits, one or more chapters from the additional topics (Chapters 8–11).

A few features distinguish this book from similar books on the subject.

- The book offers a *unique and informal treatment of logic* in Chapter 3. This allows students to experiment with mathematical thinking and writing from the beginning, and to develop problem-solving skills in a more natural and motivated way. The topic of logic is covered lightly, so that students become familiar with the logic symbols and related notions, such as negation and proof by contrapositive.
- Exercises are embedded throughout the text, requiring students to pause and think as they read it. Solutions are available at the end of each chapter.
- Every chapter ends with a Problems section, containing more advanced exercises without solutions. Overall, there are more than 350 problems in the book, most of which are *non-technical*, and require students to be creative and combine previous ideas in a novel way. Working on these problems is an integral part of the learning process, and it is crucial that students attempt them on a regular basis. It is through working on new problems

that students develop the required thinking and creative skills. Solutions to about half of these problems are provided to instructors in a solution manual, which is available at www.cambridge.org/fuchs.

- Quite often, there is a discussion on *how to approach* a problem, and the process of discovering a valid proof.
- The book touches on familiar topics, such as the quadratic formula, functions and prime numbers, but from a *more advanced and theoretical point of view*. This makes the presentation relevant and naturally bridges students' previous knowledge with a deeper and more advanced treatment of these topics.

The book is aimed at first-year students, and there are no required prerequisites other than basic high school algebra. It will require, however, persistence, maturity, and readiness perhaps to abandon old habits in studying mathematics.