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978-1-009-09628-7 — Introduction to Proofs and Proof Strategies
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Excerpt
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1 Numbers, Quadratics and Inequalities

We begin our journey by taking a closer look at some familiar notions, such as quadratic equations and inequalities. And, rather than using mechanical computations and algorithms, we focus on more fundamental questions. Where does the quadratic formula come from and how can we prove it? What are the rules that can be used with inequalities, and how can we justify them? These questions will lead us to look at a few proofs and mathematical arguments. We highlight some of the main features of a mathematical proof, and discuss the process of constructing mathematical proofs.

We also review informally the types of numbers often used in mathematics, and introduce relevant terminology.

1.1 The Quadratic Formula

The general formula for solving an equation of the form $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

was most likely presented to you in high school, and you learned how to use it for solving quadratic equations in various settings. However, if you have not seen a proof, or some sort of explanation, it would be hard to see where this formula is coming from, and why it works. In fact, the proof of this formula is quite straightforward, and requires only certain algebraic manipulations. We therefore start by properly stating a theorem on quadratic equations, and then present a proof using the method the *completing the square*.

Theorem 1.1.1 (The Quadratic Formula) *Let a, b, c be real numbers, with $a \neq 0$. The equation $ax^2 + bx + c = 0$ has*

1. *no real solutions if $b^2 - 4ac < 0$,*
2. *a unique solution if $b^2 - 4ac = 0$, given by $x = -\frac{b}{2a}$,*
3. *two distinct solutions if $b^2 - 4ac > 0$, given by*

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Remarks.

- The quantity $b^2 - 4ac$ is called the *discriminant* of the quadratic equation, and is often denoted by Δ, the capital Greek letter *Delta*. The theorem implies that the number of real solutions depends on whether $\Delta < 0$, $\Delta = 0$ or $\Delta > 0$.
- We used the terms *real numbers* and *real solutions* in the statement of the theorem. For now, let us think of real numbers as representing points on an infinite number line. A real number may be a whole number, positive or negative, a fraction, etc. We will discuss later in more detail the notion of a real number and the real number system.

Proof First, let us multiply both sides of the equation by $4a$. As $a \neq 0$, this leads to the following equivalent equation:

$$4a^2x^2 + 4abx + 4ac = 0.$$

Next, we add and subtract the term b^2 to the left-hand side:

$$4a^2x^2 + 4abx + b^2 - b^2 + 4ac = 0.$$

We now observe that the expression $4a^2x^2 + 4abx + b^2$ or, equivalently, $(2ax)^2 + 2 \cdot 2ax \cdot b + b^2$, is a perfect square. Replacing these three terms by $(2ax + b)^2$ and moving the remaining terms to the right-hand side leads to

$$(2ax + b)^2 = b^2 - 4ac.$$

The resulting equation is simpler than the original one, as the unknown x appears only once. It will be easier now to *solve for* x and obtain the quadratic formula. Nevertheless, we must be careful. Solving for x will involve using square roots, which cannot be applied to negative numbers. We therefore consider three possible cases.

1. If $b^2 - 4ac < 0$, then the equation has no real solutions, as $(2ax + b)^2 \geq 0$ for all real numbers x.
2. If $b^2 - 4ac = 0$, then the equation becomes $(2ax + b)^2 = 0$. This implies that $2ax + b = 0$, from which it follows that $x = -\frac{b}{2a}$. Consequently, the equation has a unique solution in this case.
3. If $b^2 - 4ac > 0$, then the equation has two real solutions, given by

$$2ax + b = \pm\sqrt{b^2 - 4ac}.$$

We can rearrange this equality to obtain the familiar quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

as needed. □

Exercise 1.1.2
Suppose that a, b and c are real numbers with $a > 0$ and $c < 0$. How many solutions does the quadratic equation $ax^2 + bx + c = 0$ have?

This was our first proof. Take a close look at it! What features can you identify in that proof? Here are a few important remarks.

Remarks.

- The proof included quite a few words and sentences, in natural language, and not just mathematical symbols such as equations, numbers and formulas. This will happen with most mathematical proofs. A mathematical argument should be made out of complete sentences, which may contain words, symbols, or a combination of both. The words are meant to help the reader follow the logical flow of the argument, explain the main steps, and connect the various parts of the proof. Words such as *if-then*, *and*, and *or* often appear in mathematical arguments and should be used properly. In Chapter 3 we discuss in detail the meaning of these words in mathematics, and how to use them in mathematical proofs.
- At the end of the proof, we placed the symbol □. This is a common way to denote the end of a mathematical proof or, more generally, the end of an argument. In other books you might see the symbol ■ or the acronym *Q.E.D.* used instead. The latter comes from the Latin phrase *Quod Erat Demonstrandum*, meaning "that which was to be shown." In this book we will use our square □.
- A mathematical argument is normally based on facts that have been previously validated, or agreed upon. For example, in the proof of Theorem 1.1.1, we used the identity $(x + y)^2 = x^2 + 2xy + y^2$, which is valid for every two real numbers x and y. Should we have also proved this formula? Well, we could, but we assumed that it was well established prior to proving the theorem, and so there was no need to re-explain or prove it again. This sort of judgment needs to be done each time a mathematical argument is presented to an audience, and you will need to ask yourself which facts should be well known to the reader? What other theorems or claims may I refer to in my proof? What are the main steps, or ideas, in the argument? What is the main tool (or tools) used in my proof? Should I mention them explicitly? With time and practice, you will develop your own style of writing mathematical proofs. The feedback you will get from your teachers and classmates will help you improve your writing and polish your arguments.

There are several reasons why proofs are important. First, a proof *validates* the truth of a general statement. Once a theorem is proved, it remains true forever (unless an error is found). For instance, Theorem 1.1.1 implies that a quadratic equation can never have three distinct solutions, no matter how hard you try to find one, or how much time you spend searching. This is the strength of a proof. We can say now, without a doubt, that every given quadratic equation must have zero, one or two real solutions, and there are no other options or exceptions.

Second, a proof often gives us an insight as to *why* the theorem is valid, and may suggest strategies for proving other related statements. For instance, can we prove a similar theorem on cubic equations? Quite often, discovering a proof for a given statement serves as a step in proving other related or more general results.

1.2 Working with Inequalities – Setting the Stage

In your high school years, you must have spent a substantial amount of time on equations. You had to rearrange, simplify, and solve equations regularly. However, working with inequalities can be more challenging, and one has to be much more careful with arguments and computations involving inequalities.

Example 1.2.1 Consider the equation $\frac{1}{x} = x$. Solving it is quite straightforward. We multiply both sides by x to get the equation $x^2 = 1$, which has solutions $x = 1$ and $x = -1$.

On the other hand, how would one solve the inequality $\frac{1}{x} > x$? Here, we cannot multiply both sides by x as we did previously, since the inequality sign would need to be reversed if $x < 0$. Instead, we consider two cases.

- If $x > 0$, then multiplying by x gives $1 > x^2$, and the *positive* xs satisfying this inequality are those between 0 and 1. Namely, we conclude that $0 < x < 1$.
- If $x < 0$, we get $1 < x^2$ (the inequality sign is reversed), and the *negative* xs satisfying this condition are those which are less than -1. That is, $x < -1$.

In summary, the set of real xs for which $\frac{1}{x} > x$ are the numbers between 0 and 1, and those that are smaller than -1. We can write:

$$\frac{1}{x} > x \quad \text{if and only if} \quad x < -1 \ \text{ or } \ 0 < x < 1.$$

In mathematics, the words "*if and only if*" indicate a two-sided implication. If x solves the inequality, then it must satisfy the condition "$x < -1$ or $0 < x < 1$," and if this condition is satisfied, then x solves the inequality.

This example shows some of the complications that may arise while working with inequalities, and how important it is to be able to manipulate them properly.

We begin by listing a few basic properties involving inequalities, which we temporarily refer to as *Basic Facts*.

Basic Facts. *Suppose that x, y, z are real numbers.*

1. *Exactly one of the following must occur: $x < y$, $y < x$ or $x = y$.*
2. *If $x < y$ and $y < z$ then $x < z$.*
3. *If $x < y$ then $x + z < y + z$.*
4. *If $x < y$ and $z > 0$ then $xz < yz$.*
5. *If $z > 0$, there is exactly one positive number $\sqrt{z}$, whose square is z.*

Note that the condition $a < b$ has the same meaning as $b > a$. Moreover, we allow ourselves to use symbols such as $\leq$ and $\geq$ to mean "*less than or equal to*," and "*greater than or equal to*," respectively.

For now, we accept the Basic Facts without proof. As we will discuss later, certain theories in mathematics are built on some foundational assumptions, often called *axioms*, which we accept without proof.

There are more basic properties involving inequalities. We have decided not to include them as they can be derived as consequences from the above basic facts.

Proposition 1.2.2 *For all real numbers x, y, w, z, the following hold true.*

1. *If $x < y$ and $z < 0$ then $xz > yz$.*
2. *If $x < y$ and $z < w$ then $x + z < y + w$.*
3. *If $0 < x < y$ and $0 < z < w$ then $xz < yw$.*

Proof 1. Using Basic Fact 3, we add $-z$ to the inequality $z < 0$, to get

$$z + (-z) < 0 + (-z) \quad \text{which simplifies to} \quad 0 < -z.$$

Now, we use Basic Fact 4 and multiply both sides of $x < y$ by $-z$, which gives us $(-z)x < (-z)y$ or, equivalently, $-zx < -zy$.

Finally, we add zx and zy to both sides (using Basic Fact 3 again), and get $zy < zx$, or $xz > yz$, as needed.

2. We first use Basic Fact 3 twice. Adding z to both sides of $x < y$ gives $x + z < y + z$. Adding y to both sides of $z < w$ gives $y + z < y + w$.

From Basic Fact 2 it follows that $x + z < y + z$ and $y + z < y + w$ imply $x + z < y + w$, as needed.

3. See the exercise below. □

Note again how our proofs contained words, and that complete sentences were used. If we remove all words from, say, the proof of Part 1 above, we would get something like

$$z < 0 \quad z + (-z) < 0 + (-z) \quad 0 < -z \quad x < y \quad -zx < -zy \quad xz > yz,$$

which cannot be considered a proof (even though it contains the key steps). Without words and complete sentences, it would be very hard for the reader to follow the argument, and the logic that was used. The reader may conclude that the argument is incomplete, unclear, or even flawed.

If we replace all the inequality signs $<$ and $>$ in Proposition 1.2.2 by $\le$ and $\ge$, respectively, we obtain another valid proposition, that can be proved using similar arguments.

Exercise 1.2.3
Prove Part 3 of Proposition 1.2.2.

Exercise 1.2.4
Use the Basic Facts and Proposition 1.2.2 to prove the following.

- $0 < 1$. (Hint: Show that $1 < 0$ is impossible.)
- For every non-zero real number x, we have $x^2 > 0$.

Our next proposition involves squaring and square-rooting inequalities.

Proposition 1.2.5 *Let a and b be two real numbers.*

1. *If $0 < a < b$ then $a^2 < b^2$ and $\sqrt{a} < \sqrt{b}$.*
2. *Similarly, if $0 \le a \le b$, then $a^2 \le b^2$ and $\sqrt{a} \le \sqrt{b}$.*

Exercise 1.2.6
Show that the assumption that a and b are positive is crucial. That is, show that if a and b are two real numbers, and $a < b$, then $a^2 < b^2$ might be false.

Proof of Proposition 1.2.5 We prove Part 1 only. The proof of Part 2, which is almost identical, is left to the reader.

Suppose that $0 < a < b$. As $a < b$ and $a > 0$, we can use Basic Fact 3 with $x = z = a$ and $y = b$ to get $a^2 < ab$. Similarly, as $b > 0$, we can use Basic Fact 3 again to get $ab < b^2$. Now, from $a^2 < ab$ and $ab < b^2$ we get, from Basic Fact 1, that $a^2 < b^2$.

To prove the second inequality, we rearrange the inequality $a < b$ and use the difference of squares formula $x^2 - y^2 = (x + y)(x - y)$:

$$a < b \quad \Rightarrow \quad b - a > 0 \quad \Rightarrow \quad (\sqrt{b} + \sqrt{a})(\sqrt{b} - \sqrt{a}) > 0.$$

Note that Basic Fact 5 has been used here implicitly. The symbol $\Rightarrow$ means "*implies that*," and will be discussed in detail in Chapter 3. Finally, we multiply both sides, using Basic Fact 4 with $z = \frac{1}{\sqrt{b}+\sqrt{a}}$, to get $\sqrt{b} - \sqrt{a} > 0$, or $\sqrt{a} < \sqrt{b}$, as needed. □

1.3 The Arithmetic-Geometric Mean and the Triangle Inequalities

In this section, we present two fundamental and important inequalities in mathematics: the Arithmetic-Geometric Mean Inequality and the Triangle Inequality. Both are central to many areas of mathematics and have numerous applications in physics, statistics and other sciences. Moreover, the approach we use to prove these inequalities is quite general, and can be used to prove other useful statements.

The Arithmetic-Geometric Mean Inequality

We begin with the following definition.

Definition 1.3.1 The *arithmetic mean* of two real numbers x and y is $\frac{x+y}{2}$. If $x, y \geq 0$, then their *geometric mean* is $\sqrt{x \cdot y}$.

You may be already familiar with the arithmetic mean, often called the *average* of two numbers. The geometric mean is another type of *average* which shows up frequently in various applications. Let us look at a few examples.

Example 1.3.2 The arithmetic mean of 2 and 8 is $\frac{2+8}{2} = 5$, and their geometric mean is $\sqrt{2 \cdot 8} = 4$.

The arithmetic mean of 5 and 45 is $\frac{50}{2} = 25$ and their geometric mean is $\sqrt{225} = 15$.

Note how in both cases, the arithmetic mean is greater than the geometric mean. As we will shortly see, this is not a coincidence.

The arithmetic mean of -10 and 7 is -1.5, and their geometric mean is undefined.

Example 1.3.3 A bank offers a savings account that pays interest once a year as follows. The rate for the first year is 10%, and for the second year it is 20%. For instance, if the initial investment is \$250, then after one year, this amount grows to $\$250 \cdot 1.1 = \$275$, and after two years to $\$275 \cdot 1.2 = \$330$. In general, if the initial investment is x, then after two years, it grows to $x \cdot 1.1 \cdot 1.2 = 1.32x$.

What would be a sensible way to define an "*average rate*" for the first two-year period?

Solution

We might want to look for a hypothetical fixed rate r that would lead to the same final amount. Thus, we want r to satisfy the condition $x \cdot r \cdot r = x \cdot 1.1 \cdot 1.2$ (for every value of x). We get

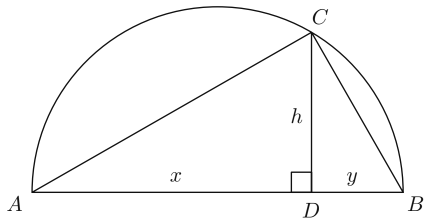

Figure 1.1 A right triangle inscribed in a (half) circle.

$$r^2 = 1.1 \cdot 1.2 \qquad \Rightarrow \qquad r = \sqrt{1.1 \cdot 1.2} = \sqrt{1.32} \approx 1.1489.$$

We conclude that the average interest rate is about 14.89% (and *not* 15%). Note that 1.1489 is the *geometric mean* of 1.1 and 1.2.

Example 1.3.4 In Figure 1.1, AB is a diameter of a circle, and CD is perpendicular to AB. If x, y and h are the lengths of AD, BD and CD, respectively, how can we express h in terms of x and y?

Solution

One way to proceed, is to observe that triangles ADC, CDB and ACB are all right triangles (remember that inscribed angles in a circle, subtended by a diameter, are right angles). Therefore, we can apply the Pythagorean Theorem to get

$$AC^2 = AD^2 + DC^2, \quad CB^2 = CD^2 + DB^2 \quad \text{and} \quad AB^2 = AC^2 + CB^2.$$

We now use the first two equalities to replace AC^2 and CB^2 by $AD^2 + DC^2$ and $CD^2 + DB^2$ in the third equality:

$$AB^2 = (AD^2 + DC^2) + (CD^2 + DB^2) = AD^2 + 2CD^2 + DB^2.$$

Expressing all quantities in terms of x, y and h leads to

$$(x+y)^2 = x^2+2h^2+y^2 \quad \Rightarrow \quad x^2+2xy+y^2 = x^2+2h^2+y^2 \quad \Rightarrow \quad h^2 = xy,$$

which gives $h = \sqrt{xy}$. In other words, h is the *geometric mean* of x and y.

Exercise 1.3.5

Use similar triangles instead of the Pythagorean Theorem, to provide an alternative solution to Example 1.3.4.

We are now ready to present the Arithmetic-Geometric Mean Inequality.

Theorem 1.3.6 (The Arithmetic-Geometric Mean Inequality) *For every two real numbers x and y, we have $x \cdot y \leq \left(\frac{x+y}{2}\right)^2$, and equality holds if and only if $x = y$.*

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If, in addition, $x \geq 0$ *and* $y \geq 0$*, then* $\sqrt{x \cdot y} \leq \frac{x+y}{2}$.

The last statement says that when $x, y \geq 0$, the geometric mean is less than or equal to the arithmetic mean of x and y.

Let us first make sure that we fully understand the statement, and what needs to be proved. Keeping in mind that "*if and only if*" means a double-sided implication, we see that there are three statements included in the first sentence.

1. For all real numbers x and y, $x \cdot y \leq \left(\frac{x+y}{2}\right)^2$.
2. If $x = y$, then $x \cdot y = \left(\frac{x+y}{2}\right)^2$.
3. If $x \cdot y = \left(\frac{x+y}{2}\right)^2$, then $x = y$.

The second sentence in Theorem 1.3.6 also needs to be proved, but this will follow by applying square roots to both sides of $x \cdot y \leq \left(\frac{x+y}{2}\right)^2$.

Let us start by focusing on Part 1. How would one prove such an inequality for all x and y? We cannot substitute numbers for x and y, as our argument must be completely general. But, as a start, we can try to rewrite the given inequality, with the hope of simplifying it to an inequality that would be easier to prove. We call this kind of work "rough work," since we are not writing an actual proof yet, but only doing preliminary experimentation to try and *discover a proof*.

Rough Work

$$x \cdot y \leq \left(\frac{x+y}{2}\right)^2 \quad \Rightarrow \quad x \cdot y \leq \frac{x^2+2xy+y^2}{4} \quad \Rightarrow \quad 4xy \leq x^2+2xy+y^2$$

$$\Rightarrow \quad 0 \leq x^2-2xy+y^2 \quad \Rightarrow \quad 0 \leq (x-y)^2.$$

Can this rough work be considered a proof? No. First, there are no words and full sentences explaining the argument. More importantly, a proof cannot begin with the statement that needs to be proved. Remember, we cannot assume the validity of the inequality $x \cdot y \leq \left(\frac{x+y}{2}\right)^2$. Our task is to provide a proof that validates the inequality. We may only use facts that are known to be true, such as elementary high school algebra.

However, we did achieve something. Using algebraic manipulations, we were able to obtain a simpler inequality, namely $0 \leq (x-y)^2$, which holds true for all x and y, by Exercise 1.2.4. We might be able to use it as our starting point, and work backwards in the rough work. If we manage to reverse all the steps, we will end up with the desired inequality, and that would be our proof.

We are now ready to prove the theorem.

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Proof of Theorem 1.3.6 1. For every two real numbers x and y, we have $0 \leq (x-y)^2$, as $a^2 \geq 0$ for every real number a. We expand and add $4xy$ to both sides, and get

$$0 \leq (x-y)^2 \quad \Rightarrow \quad 0 \leq x^2-2xy+y^2 \quad \Rightarrow \quad 4xy \leq x^2+2xy+y^2.$$

We divide by 4, and notice that the right-hand side is a perfect square:

$$xy \leq \frac{x^2+2xy+y^2}{4} \quad \Rightarrow \quad xy \leq \left(\frac{x+y}{2}\right)^2.$$

We conclude that the last inequality is valid for all real numbers x and y, as needed.

2. To prove this part, we do not need rough work. We can simply replace y by x and verify that we get an equality. Indeed, if $x = y$, then the left-hand side becomes $xy = x^2$, and the right hand side becomes

$$\left(\frac{x+y}{2}\right)^2 = \left(\frac{2x}{2}\right)^2 = x^2.$$

We have proved that when $x = y$ we have an equality, as needed.

3. Finally, we assume that $xy = \left(\frac{x+y}{2}\right)^2$, and prove that $x = y$. This can be done by simplifying the former equality, with the hope that, at some point, it will become clear that x and y must be equal to each other. The steps we follow resemble the rough work:

$$x \cdot y = \left(\frac{x+y}{2}\right)^2 \quad \Rightarrow \quad x \cdot y = \frac{x^2+2xy+y^2}{4} \quad \Rightarrow \quad 4xy = x^2+2xy+y^2$$

$$\Rightarrow \quad 0 = x^2-2xy+y^2 \quad \Rightarrow \quad 0 = (x-y)^2.$$

The only number that squares to zero is 0, and so $x - y = 0$, from which we conclude that $x = y$, as needed.

To prove the second sentence in the theorem, suppose that $x, y \geq 0$. We already know that the inequality

$$xy \leq \left(\frac{x+y}{2}\right)^2$$

holds true. Using Proposition 1.2.5, we apply square roots to both sides, and get

$$\sqrt{xy} \leq \frac{x+y}{2},$$

which concludes our proof. □

Here is an example in which the Arithmetic-Geometric Mean Inequality is used.