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Stacks Project Expository Collection (SPEC)

The Stacks Project Expository Collection (SPEC) compiles expository articles in advanced algebraic geometry that are intended to bring graduate students and researchers up to speed on recent developments in the geometry of algebraic spaces and algebraic stacks. The articles in the text make explicit in modern language many results, proofs, and examples that were previously only implicit, incomplete, or expressed in classical terms in the literature. Where applicable this is done by explicitly referring to the Stacks project for preliminary results. Topics include the construction and properties of important moduli problems in algebraic geometry (such as the Deligne–Mumford compactification of the moduli of curves, the Picard functor, or moduli of semistable vector bundles and sheaves) and arithmetic questions for fields and algebraic spaces.

PIETER BELMANS is Assistant Professor of Mathematics at the University of Luxembourg. He studies algebraic geometry and noncommutative algebra from the point of view of derived categories. He developed the infrastructure that runs the Stacks project.

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Preface

The *Stacks project* is "an open source textbook and reference work on algebraic geometry" that has been consistently growing since 2008. At the time of writing, the material encompasses more than 7500 pages, and it is regularly expanded. It contains background material in commutative algebra and algebraic geometry, advanced material, and even previously unpublished work. Widely used as a reference in algebraic geometry, the Stacks project is most easily accessed via its website

`https://stacks.math.columbia.edu`

The project is collaborative, with more than 500 contributors, listed on the website. Every submission and comment is reviewed carefully, for correctness and coherence with the rest of the text, by the second editor of this volume.

Although the Stacks project covers most basic and some advanced algebraic geometry, many interesting and useful advanced topics are not yet part of the Stacks project. To add a new topic often requires a large amount of work, partially due to the requirement that all prerequisites are built from scratch. Moreover, for some subjects, the existing literature may not use the newest foundations or machinery available to mathematicians now. Before blindly adding a new result or concept, it thus makes sense to explore it and find efficient proof strategies using, as much as possible, already existing material in the Stacks project. The chapters in this volume are first and foremost expository presentations in their own right. But they can also be considered as explorations on subjects which deserve inclusion in the Stacks project; we hope they can someday be used as the initial versions of new chapters of the Stacks project.

Each chapter grew out of group work at one of two workshops affiliated with the Stacks project that we organized. The first workshop took place in person at the University of Michigan in summer 2017, and the second – although originally intended to take place in Ann Arbor again – became a virtual workshop during summer 2020. At each of these workshops, senior algebraic geometers led groups of graduate students and postdoctoral scholars in learning and exploring advanced topics in algebraic geometry, with the goal of writing careful expositions at a level appropriate for advanced graduate students. We hope that this book will serve as a useful reference for students and researchers interested in learning about these topics. While most of the chapters are primarily expository, many contain new examples and proofs not found elsewhere in the literature.

We now give a brief description of the chapters in this volume. The first several are related to moduli problems in algebraic geometry. In Chapter 1, "Projectivity of the moduli of curves", following a method of Kollár that avoids geometric invariant theory, the authors give an exposition of the projectivity (over $\operatorname{Spec}\mathbb{Z}$) of the Deligne–Mumford moduli space $\overline{M}_g$ of stable curves of genus $g \geq 2$. Chapter 2, "The stack of admissible covers is algebraic", gives a proof of the algebraicity of the stack of classifying stable genus-g curves equipped with "admissible" G-covers, following a result of Abramovich, Corti, and Vistoli (and going back to ideas of Harris and Mumford); this chapter also has an exposition of group actions on algebraic spaces. Chapter 3, "Projectivity of the moduli space of vector bundles on a curve", explains a proof that the moduli space of semistable vector bundles (of fixed rank and degree) on a curve of genus $g \geq 2$ is projective; unlike the classical proof via geometric invariant theory, the proof here relies on the modern notion of good moduli spaces and uses a method due to Esteves and Popa to prove the projectivity.

Going from bundles on curves to sheaves on varieties, in Chapter 4, "Boundedness of semistable sheaves", the authors explain, following work of Langer, why the moduli space of semistable torsion-free sheaves with fixed Hilbert polynomial is bounded for any projective variety in any characteristic; the key result is an upper bound for the maximal slope of the restriction of the sheaf under consideration to a general hypersurface. Chapter 5, "Theorem of the Base", contains a modern proof of a fundamental theorem in algebraic geometry: the Néron–Severi group of a proper variety is finitely generated. The proof relies on a Weil cohomology (such as ℓ-adic cohomology) to prove finite generation up to torsion, and a reduction to the smooth projective case via de Jong's alterations to handle torsion. In Chapter 6, "Weil restriction for schemes and beyond", the authors discuss, with several examples, the notion of Weil restriction for both schemes and algebraic spaces; amongst other things, one finds a proof of the preservation of quasi-projectivity under this operation. Chapter 7, "Heights over finitely generated fields", contains an exposition of several notions of heights in modern arithmetic geometry: naive heights over number fields, geometric heights over function fields, and finally Moriwaki heights over finitely generated fields.

Chapter 8, "An explicit self-duality", gives an explicit construction of a consequence of Grothendieck duality: if $A \to B$ is a finite flat complete intersection map with A noetherian and B local, then B is self-dual as an A-module (and canonically so after choosing a presentation). In Chapter 9, "Tannakian reconstruction of coalgebroids", the author gives an exposition of a recent result of Schäppi generalizing classical Tannakian reconstruction theorems: there is an explicit criterion allowing one to reconstruct a group scheme G over a commutative ring R from its category of representations of finite projective R-modules.

We thank the authors for their contributions to this volume. We also thank the mentors (Jarod Alper, Bhargav Bhatt, Brian Conrad, Matthew Emerton, Max Lieblich, Davesh Maulik, Martin Olsson, Alex Perry, Ravi Vakil, Kirsten Wickelgren) and the participants of the Stacks project workshops for making these workshops such pleasant and productive events, and we thank the National Science Foundation, Compositio, and the University of Michigan for funding these workshops. Finally, we thank the contributors to and users of the Stacks project.

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