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Projectivity of the moduli of curves

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Abstract In this expository paper, we show that the Deligne–Mumford moduli space of stable curves is projective over $\mathrm{Spec}(\mathbf{Z})$. The proof we present is due to Kollár. Ampleness of a line bundle is deduced from nefness of a related vector bundle via the ampleness lemma, a classifying map construction. The main positivity result concerns the pushforward of relative dualizing sheaves on families of stable curves over a smooth projective curve.

Introduction

Let $\overline{\mathcal{M}}_g$ be the moduli stack of stable curves of genus $g \geq 2$ and write $\overline{M}_g$ for its corresponding moduli space. We prove that the moduli of stable curves is projective in the following sense, see Theorem 1.7.2:

Theorem *The Deligne–Mumford moduli space $\overline{M}_g$ of stable curves of genus $g \geq 2$ is a projective scheme over* $\mathrm{Spec}(\mathbf{Z})$.

In particular, this means that $\overline{M}_g$, which is *a priori* just an algebraic space, is actually a projective scheme over $\mathbf{Z}$. Together with the work of Deligne and Mumford [9] (see also [30, Theorem 0E9C]) this means that $\overline{M}_g$ is actually an irreducible projective scheme over $\mathbf{Z}$.

We explain a proof due to Kollár in [21]. Specifically, the task of showing that a certain line bundle on $\overline{M}_g$ is ample is transferred, via

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Kollár's ampleness lemma, to the problem of showing that a related vector bundle is nef on $\overline{M}_g$. Since nefness is a condition that only depends on the behaviour of the vector bundle upon restriction to curves, projectivity is thus reduced to a problem regarding positivity of one-parameter families of stable curves.

Kollár's method differs from other existing proofs of projectivity of $\overline{M}_g$ in at least two main ways: First, the technique is independent of the methods of geometric invariant theory (GIT), on which the proofs of [29, 11, 7] rely. In a similar spirit, in [1], which is Chapter 3 of this volume, the projectivity of the moduli space of semistable vector bundles on a curve is established without using GIT.

Second, Kollár's criterion does not require one to directly check that a line bundle on the moduli space is ample, in contrast to the approach of Knudsen and Mumford [19, 17, 18]; rather, one only needs to show that some vector bundle on the moduli space is nef. As such, this method has since been used in other settings, such as in the moduli of weighted stable curves [14], of stable varieties [22], and, recently, of K-polystable Fano varieties [6, 33].

An outline of this article is as follows. We set up notation in regards to the moduli of curves in Section 1.1, after which we begin in Sections 1.2–1.4 with some material on positivity of sheaves. In Section 1.5, we explain Kollár's ampleness lemma; see Proposition 1.5.4. In Section 1.6, we prove the main positivity statement: the pushforward of the relative dualizing sheaf of a 1-parameter family of stable curves of genus at least 2 is nef; see Theorem 1.6.10. Finally, we put everything together in Section 1.7 to show that $\overline{M}_g$ is projective over $\mathbf{Z}$ when $g \geq 2$.

Conventions Throughout, k will denote a field. Following the conventions of the Stacks project, a *variety* is a separated integral scheme of finite type over a field k and a *curve* is a variety of dimension 1, see [30, Definitions 020D and 0A23]. Given a scheme X over k and a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, we write

$$h^i(X, \mathcal{F}) := \dim_k(H^i(X, \mathcal{F})) \quad \text{for all } i \in \mathbf{Z}.$$

1.1 Stable curves

In this section, we record the definition of the moduli problem in which we are primarily interested, namely that of the moduli space of stable curves. The main references are [9] and [30, Chapter 0DMG].

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First we define what we mean by a family of curves. Compare the following with [30, Situation 0D4Z], and with [30, Definitions 0C47, 0C5A, and 0E75]. We diverge slightly from the Stacks project in that we require our families of nodal curves to have geometrically connected fibres. Caution: the closed fibres of a family of nodal curves are *not* curves in the sense of our conventions, as they may be reducible. See [30, Section 0C58] for a discussion on such terminology.

Definition 1.1.1 Let S be a scheme.

(i) A *family of nodal curves over* S is a flat, proper, finitely presented morphism of schemes $f\colon X \to S$ of relative dimension 1 such that all geometric fibres are connected and smooth except at possibly finitely many nodes.
(ii) A *family of stable curves over* S is a family of nodal curves such that the geometric fibres have arithmetic genus ≥ 2 and do not contain rational tails or bridges.
(iii) A family of stable curves over S is said to *have genus* g if all geometric fibres have genus g.

Condition (ii) is equivalent to ampleness of the dualizing sheaf, and also finiteness of automorphism groups. See [30, Section 0E73] for details. For the following, see [30, Definition 0E77].

Definition 1.1.2 For $g \geq 2$, the *moduli stack of stable curves of genus* g is the category $\overline{\mathcal{M}}_g$ fibred in groupoids whose category of sections over a scheme S has objects given by families of stable curves of genus g over S, and morphisms given by isomorphisms of families over S.

The stack $\overline{\mathcal{M}}_g$ is a smooth, proper Deligne–Mumford stack over $\operatorname{Spec}(\mathbf{Z})$; see [30, Theorem 0E9C]. Classically, and in many geometric applications such as [13], it is convenient to work with a space rather than the stack. As such, it is useful to extract an algebraic space which is, in some sense, the closest approximation of the stack, obtained by "forgetting" the automorphism groups: this is the notion of a *uniform categorical moduli space* or simply a *moduli space* of a stack; see [30, Definition 0DUG].

Lemma 1.1.3 *The stack* $\overline{\mathcal{M}}_g$ *admits a uniform categorical moduli space* $f_g\colon \overline{\mathcal{M}}_g \to \overline{M}_g$ *such that* f_g *is separated, quasi-compact, and a universal homeomorphism.*

Proof The stack $\overline{\mathcal{M}}_g$ has finite inertia by [30, Lemmas 0E7A and

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0DSW], so the existence of f_g follows from the Keel–Mori theorem [30, Theorem 0DUT]. □

Definition 1.1.4 The space $\overline{M}_g$ is the *moduli space of curves of genus* g.

Our primary goal is to show that $\overline{M}_g$ is projective over $\mathbf{Z}$; see Theorem 1.7.2. Thus we must exhibit an ample invertible sheaf on $\overline{M}_g$. We obtain invertible sheaves on the moduli space by taking powers of invertible sheaves on the stack $\overline{\mathcal{M}}_g$, via the following general fact:

Lemma 1.1.5 *Let $\mathcal{X}$ be an algebraic stack. Assume the inertia $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is finite and let $f\colon \mathcal{X} \to M$ be its moduli space, as in [30, Theorem 0DUT]. Then*

$$f^*\colon \mathrm{Pic}(M) \to \mathrm{Pic}(\mathcal{X})$$

is injective. If $\mathcal{X}$ is furthermore quasi-compact, then the cokernel of f^* is annihilated by a positive integer.

Proof For the injectivity, note that $f_*\mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_M$ as M is initial for morphisms from $\mathcal{X}$ to algebraic spaces and the structure sheaf represents the functor $\mathrm{Hom}(-, \mathbf{A}^1)$. Thus if $\mathcal{N} \in \mathrm{Pic}(M)$ is such that $f^*\mathcal{N} \cong \mathcal{O}_{\mathcal{X}}$, the canonical map $\mathcal{N} \to f_*f^*\mathcal{N} \to \mathcal{O}_M$ is an isomorphism as $\mathcal{N}$ is locally trivial. This further shows that if $\mathcal{N}_1, \mathcal{N}_2 \in \mathrm{Pic}(M)$ are such that there exists an isomorphism $\varphi\colon f^*\mathcal{N}_1 \to f^*\mathcal{N}_2$, then there is a unique isomorphism $\psi\colon \mathcal{N}_1 \to \mathcal{N}_2$ such that $f^*\psi = \varphi$.

We now show that, if $\mathcal{X}$ is furthermore quasi-compact, then there is a positive integer n such that, for every $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$, $\mathcal{L}^{\otimes n} \cong f^*\mathcal{N}$ for some $\mathcal{N} \in \mathrm{Pic}(M)$. For this, we may replace $\mathcal{X}$ by any $\mathcal{X}'$ with a surjective separated étale morphism $h\colon \mathcal{X}' \to \mathcal{X}$ of algebraic stacks inducing isomorphisms on automorphism groups. Indeed, [30, Lemma 0DUV] gives the cartesian square

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ M' & \longrightarrow & M \end{array}$$

where M' is the moduli space of $\mathcal{X}'$. If there were $\mathcal{N}' \in \mathrm{Pic}(M')$ such that $h^*\mathcal{L}^{\otimes n} \cong f'^*\mathcal{N}'$, then the injectivity of $f'^*\colon \mathrm{Pic}(M') \to \mathrm{Pic}(\mathcal{X}')$ shows that the étale descent datum for $h^*\mathcal{L}^{\otimes n}$ over $\mathcal{X}$ induces an étale descent datum for $\mathcal{N}'$ over M, yielding $\mathcal{N} \in \mathrm{Pic}(M)$ as above.

Choose such a cover $h\colon \mathcal{X}' \to \mathcal{X}$ as in [30, Lemma 0DUE]: $\mathcal{X}' = \coprod_{i \in I} \mathcal{X}_i$ where each $\mathcal{X}_i$ is a quotient stack $[U_i/R_i]$, $(U_i, R_i, s_i, t_i, c_i)$ is

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a groupoid scheme with U_i and R_i affine, and $s_i, t_i \colon R_i \to U_i$ are finite locally free of some constant rank; see [30, Lemmas 0DUM and 03BI]. Since $\mathcal{X}$ is quasi-compact, we are reduced to the case where $\mathcal{X}$ is a finite disjoint union of such stacks $\mathcal{X}_i$. Let $f_i \colon \mathcal{X}_i \to M_i$ be the moduli space. If there exists a positive integer n_i annihilating the cokernel of f_i^*, then the least common multiple n of the n_i annihilates the cokernel of f^*.

Thus it suffices to consider the case where $\mathcal{X} = [U/R]$ is as above. By [30, Proposition 06WT], an invertible $\mathcal{O}_{\mathcal{X}}$-module may be represented as a pair $(\mathcal{L}, \alpha)$ consisting of an invertible $\mathcal{O}_U$-module $\mathcal{L}$ together with an isomorphism $\alpha \colon t^*\mathcal{L} \to s^*\mathcal{L}$ of $\mathcal{O}_R$-modules as in [30, Definition 03LI]. We claim that if n is the rank of the morphisms $s, t \colon R \to U$, then $(\mathcal{L}^{\otimes n}, \alpha^n)$ is in the image of f^*. Namely, writing $\pi \colon U \to M$, there exists an invertible $\mathcal{O}_M$-module $\mathcal{N}$ and an isomorphism of invertible modules $(\pi^*\mathcal{N}, \alpha_{\mathrm{can}}) \cong (\mathcal{L}^{\otimes n}, \alpha^n)$ on the groupoid (U, R, s, t, c), where α_{can} is the identity map; this makes sense since $\pi \circ t = \pi \circ t$ as maps $R \to M$.

Construct $\mathcal{N}$ as follows. First, if $U = \bigcup U_i$ is any affine open cover, then the $V_i := \pi(U_i)$ together form an affine open cover of M. That the V_i form an open cover follows from the fact that π is the composition of the faithfully flat and finitely presented morphism $U \to \mathcal{X}$ and the universal homeomorphism $\mathcal{X} \to M$; see [30, Lemmas 01UA and 0DUP]. That the V_i are affine is because π is integral; see [30, Lemmas 03BJ and 05YU]. Next, since $t \colon R \to U$ is finite locally free, [30, Lemma 0BCY] constructs an invertible $\mathcal{O}_U$-module $\mathcal{L}' := \mathrm{Norm}_t(s^*\mathcal{L})$ as follows. Let $(\{U_i\}, \{u_{ij}\})$ be a system of cocycles locally defining $\mathcal{L}$, so that $U = \bigcup U_i$ is an affine open cover and $u_{ij} \in \mathcal{O}_U^*(U_i \cap U_j)$ are units. Then $\mathcal{L}'$ is defined by the cocycles $(\{U_i\}, \{u'_{ij}\})$ with $u'_{ij} := \mathrm{Norm}_{t^\sharp}(s^\sharp(u_{ij}))$. Finally, setting $V_i := \pi(U_i)$, [30, Lemma 03BH] implies that the u'_{ij} lie in the subgroup $\mathcal{O}_M^*(V_i \cap V_j) \subseteq \mathcal{O}_U^*(U_i \cap U_j)$ of R-invariant units, so $(\{V_i\}, \{u'_{ij}\})$ forms a system of cocycles on M defining an invertible module $\mathcal{N}$.

On the one hand, the construction implies $\mathcal{L}' \cong \pi^*\mathcal{N}$. On the other hand, [30, Lemma 0BCZ] yields an isomorphism

$$\mathrm{Norm}_t(\alpha) \colon \mathcal{L}^{\otimes n} \cong \mathrm{Norm}_t(t^*\mathcal{L}) \to \mathrm{Norm}_t(s^*\mathcal{L}) = \mathcal{L}' \cong \pi^*\mathcal{N}.$$

Thus it suffices to show that the diagram of isomorphisms

$$\begin{array}{ccc} t^*\mathcal{L}^{\otimes n} & \xrightarrow[\alpha^n]{} & s^*\mathcal{L}^{\otimes n} \\ {\scriptstyle t^*\mathrm{Norm}_t(\alpha)}\big\downarrow & & \big\downarrow{\scriptstyle s^*\mathrm{Norm}_t(\alpha)} \\ t^*\pi^*\mathcal{N} & \overset{\alpha_{\mathrm{can}}}{=\!=} & s^*\pi^*\mathcal{N} \end{array}$$

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is commutative. By properties of the norm, the compatibilities of α from [30, Definition 03LH(1)], and the diagram of [30, Lemma 03BH], we have

$$\begin{aligned}\alpha^n &= \mathrm{Norm}_c(c^*\alpha) = \mathrm{Norm}_c(\mathrm{pr}_1^*\,\alpha \circ \mathrm{pr}_0^*\,\alpha)\\ &= \mathrm{Norm}_c(\mathrm{pr}_1^*\,\alpha) \circ \mathrm{Norm}_c(\mathrm{pr}_0^*\,\alpha) = s^*\,\mathrm{Norm}_s(\alpha) \circ t^*\,\mathrm{Norm}_t(\alpha).\end{aligned}$$

Since $s = t \circ i$ where $i\colon R \to R$ is the inverse, $\mathrm{Norm}_s(\alpha) = \mathrm{Norm}_t(i^*\alpha)$. Therefore

$$s^*\,\mathrm{Norm}_t(\alpha) \circ \alpha^n \circ t^*\,\mathrm{Norm}_t(\alpha)^{-1} = s^*(\mathrm{Norm}_t(\alpha \circ i^*\alpha)).$$

This is the identity since, by [30, Lemma 077Q], $i^*\alpha$ is the inverse of α. □

We now specify some invertible sheaves on $\overline{\mathcal{M}}_g$. By [30, Definition 06TR and Lemma 06WI], the data of such a sheaf $\mathcal{L}$ are the following: for each family of stable curves $X \to S$, an invertible $\mathcal{O}_S$-module $\mathcal{L}(X \to S)$, and, for every cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S, \end{array}$$

an isomorphism of invertible $\mathcal{O}_{S'}$-modules

$$\varphi_g\colon g^*\mathcal{L}(X \to S) \cong \mathcal{L}(X' \to S')$$

such that for every composition of cartesian squares

$$\begin{array}{ccccc} X'' & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ S'' & \xrightarrow{h} & S' & \xrightarrow{g} & S \end{array}$$

the isomorphisms are subject to the cocycle condition

$$\begin{array}{ccc} h^*(g^*\mathcal{L}(X \to S)) & \xrightarrow[h^*\varphi_g]{} & h^*\mathcal{L}(X' \to S') \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle \varphi_h} \\ (gh)^*\mathcal{L}(X \to S) & \xrightarrow{\varphi_{gh}} & \mathcal{L}(X'' \to S''). \end{array}$$

Definition 1.1.6 For each integer $m \geq 1$, define an invertible sheaf λ_m on $\overline{\mathcal{M}}_g$ as follows. Given a family of stable curves $f\colon X \to S$, let $\omega_{X/S}^{\otimes m}$ be its relative dualizing sheaf; see [30, Definition 0E6Q]. This is

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an invertible $\mathcal{O}_X$-module. Note that the sheaves $f_*\omega_{X/S}^{\otimes m}$ are locally free on S. Set

$$\lambda_m(f\colon X \to S) := \det(f_*\omega_{X/S}^{\otimes m}).$$

Given a cartesian square as above, we have isomorphisms φ_g given by

$$g^* \det(f_*\omega_{X/S}^{\otimes m}) \cong \det(g^* f_*\omega_{X/S}^{\otimes m}) \to \det(f'_* g'^* \omega_{X/S}^{\otimes m}) \cong \det(f'_*\omega_{X'/S'}^{\otimes m}),$$

the functorial base change maps, and the fact that the formation of $\omega_{X/S}$ commutes with arbitrary base change; see [30, Lemma 0E6R]. Functoriality ensures that these satisfy the required cocycle condition.

Our goal will be to show that there is some m such that λ_m descends to an ample invertible sheaf on $\overline{M}_g$.

1.2 Nakai–Moishezon criterion for ampleness

In this section, we discuss the Nakai–Moishezon criterion for ampleness, relating the ampleness of an invertible sheaf with the positivity of intersection numbers. We directly prove the criterion for proper algebraic spaces over a field in Proposition 1.2.4 (compare with [21, Theorem 3.11]); the proof closely follows that of [16, Section III.1, Theorem 1], with suitable modifications. Using [30, Lemma 0D3A], one can also formulate a relative version; see, for example, [15, Proposition 2.10].

In the following, we work with proper algebraic spaces over a field. For generalities on algebraic spaces, see [30, Part 0ELT].

We will use numerical intersection theory on spaces as developed in [30, Section 0DN3]; see also [30, Section 0BEL] and [26, Section 1.1.C] for the situation of varieties. The main construction is the *intersection number* $(\mathcal{L}_1 \cdots \mathcal{L}_d \cdot Z)$ between a closed subspace $\iota\colon Z \to X$ of positive dimension d and invertible $\mathcal{O}_X$-modules $\mathcal{L}_1, \ldots, \mathcal{L}_d$: this is the coefficient of $n_1 \cdots n_d$ in the numerical polynomial

$$\chi(X, \iota_*\mathcal{O}_Z \otimes \mathcal{L}_1^{\otimes n_1} \otimes \cdots \otimes \mathcal{L}_d^{\otimes n_d}) = \chi(Z, \mathcal{L}_1^{\otimes n_1} \otimes \cdots \otimes \mathcal{L}_d^{\otimes n_d}|_Z).$$

See [30, Definition 0EDF].

The Nakai–Moishezon criterion relates ampleness to the positivity of intersection numbers. To formulate this succinctly, we make a definition. In the following, recall that a separated algebraic space Z is integral if and only if it is reduced and $|Z|$ is irreducible; see [30, Definition 0AD4] and [30, Section 03I7].

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Definition 1.2.1 Let X be a proper algebraic space over k and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We say that $\mathcal{L}$ *has positive degree* if, for every integral closed subspace Z of X of positive dimension d, $(\mathcal{L}^d \cdot Z) > 0$.

Note that the Stacks project only defines the degree of an invertible sheaf $\mathcal{L}$ either when $\mathcal{L}$ is ample or when $\dim(X) \leq 1$; see [30, Definitions 0BEW and 0AYR]. The content of the Nakai–Moishezon criterion is that if $\mathcal{L}$ has positive degree, then $\mathcal{L}$ is ample. Thus this is *a fortiori* compatible with the conventions of the Stacks project.

The main technical property we need is the permanence of positivity under finite morphisms.

Lemma 1.2.2 *Let X be a proper algebraic space over k. Let $f: Y \to X$ be a finite morphism of algebraic spaces. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. If $\mathcal{L}$ has positive degree, then $f^*\mathcal{L}$ has positive degree.*

Proof This follows from the compatibility of numerical intersection numbers and pullbacks: if $Z \subset Y$ is a proper integral closed subspace of dimension d, then

$$(f^*\mathcal{L}^d \cdot Z) = \deg(Z \to f(Z))(\mathcal{L}^d \cdot f(Z))$$

where $\deg(Z \to f(Z))$ is positive as f is finite; see [30, Lemma 0EDJ]. □

The following is the core of the inductive proof of the criterion:

Lemma 1.2.3 *Let X be a proper algebraic space over k and let D be an effective Cartier divisor of X. If $\mathcal{O}_X(D)|_D$ is ample, then $\mathcal{O}_X(mD)$ is globally generated for all $m \gg 0$.*

Proof For each $m \geq 0$, there is a short exact sequence

$$0 \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_X(mD) \to \mathcal{O}_X(mD)|_D \to 0.$$

Since $\mathcal{O}_X(D)|_D$ is ample, Serre vanishing [30, Lemma 0GFA] gives an integer m_1 such that $H^1(D, \mathcal{O}_X(mD)|_D) = 0$ for $m \geq m_1$. Hence the maps

$$\rho_m : H^1(X, \mathcal{O}_X((m-1)D)) \to H^1(X, \mathcal{O}_X(mD)),$$

arising from the long exact sequence on cohomology, are surjective for all $m \geq m_1$, yielding a nonincreasing sequence of nonnegative integers

$$h^1(X, \mathcal{O}_X(mD)) \geq h^1(X, \mathcal{O}_X((m+1)D)) \geq \cdots .$$

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There is some $m_2 \geq m_1$ after which the sequence stabilizes, so that, for all $m \geq m_2$, the ρ_m are bijective and the restriction maps

$$H^0(X, \mathcal{O}_X(mD)) \to H^0(D, \mathcal{O}_X(mD)|_D)$$

are surjective. Finally, since $\mathcal{O}_X(D)|_D$ is ample, there exists some m_3 such that $\mathcal{O}_X(mD)|_D$ is generated by its global sections for all $m \geq m_3$.

Let $m_0 := \max(m_2, m_3)$. We show that the evaluation maps

$$H^0(X, \mathcal{O}_X(mD)) \otimes_k \mathcal{O}_X \to \mathcal{O}_X(mD)$$

are surjective for all $m \geq m_0$. We verify this on stalks. For $x \in |X \setminus D|$, a global section defining mD restricts to a unit in $\mathcal{O}_X(mD)_x$ and thus generates the stalk. So consider $x \in |D|$ and let $\kappa(x)$ be the residue field of D at x; see [30, Definition 0EMW]. Since $D \to X$ is a monomorphism, $\kappa(x)$ is also the residue field at x of X by [30, Lemma 0EMX]. Consider the diagram

$$\begin{array}{ccc} H^0(X, \mathcal{O}_X(mD)) \otimes_k \kappa(x) & \longrightarrow & \mathcal{O}_X(mD) \otimes_{\mathcal{O}_X} \kappa(x) \\ \downarrow & & \downarrow \simeq \\ H^0(D, \mathcal{O}_X(mD)|_D) \otimes_k \kappa(x) & \twoheadrightarrow & \mathcal{O}_X(mD)|_D \otimes_{\mathcal{O}_D} \kappa(x) \end{array}$$

obtained from the evaluation and restriction maps upon taking the fibre at x. By our choice of m_0, the restriction map on the left is surjective and $\mathcal{O}_X(mD)|_D$ is globally generated, so the bottom map is surjective. Since the right map is an isomorphism, commutativity of the diagram implies that the top map is surjective. Nakayama's lemma then implies that the evaluation map is surjective on the local ring $\mathcal{O}_X(mD)_x$. Hence the evaluation map is surjective, meaning $\mathcal{O}_X(mD)$ is globally generated. □

Proposition 1.2.4 (Nakai–Moishezon criterion) *Let X be a proper algebraic space over k. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then $\mathcal{L}$ is ample on X if and only if $\mathcal{L}$ has positive degree.*

Proof If $\mathcal{L}$ is ample, then X is a scheme, $\mathcal{L}$ is ample in the schematic sense, and $\mathcal{L}$ has positive degree; see [30, Lemmas 0D32 and 0BEV].

Assuming $\mathcal{L}$ has positive degree, we show it is ample. We proceed by induction on $\dim(X)$. When $\dim(X) = 0$, since X is separated it is a scheme by [30, Theorem 086U], in which case the result is clear. When $\dim(X) = 1$, our assumption simplifies to $\deg(\mathcal{L}) > 0$. Now apply [30, Proposition 09YC] to obtain a finite surjective morphism $f: Y \to X$ from a scheme Y. Lemma 1.2.2 shows that $\deg(f^*\mathcal{L}) > 0$ and so [30, Lemma 0B5X] gives the ampleness of $f^*\mathcal{L}$. Since f is finite, [30, Lemma

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0GFB] shows $\mathcal{L}$ is also ample. So we assume that $\dim(X) \geq 2$ and that the criterion holds for all proper spaces over k of lower dimension.

Step 1 Using [30, Lemmas 0GFB, 0GFA], and Lemma 1.2.2, we may replace X by the reduction of an irreducible component and $\mathcal{L}$ by its restriction to assume that X is integral.

Step 2 We show that some power of $\mathcal{L}$ is effective. As X is integral, the discussion of [30, Section 0ENV] shows that $\mathcal{L}$ has a regular meromorphic section s. Consider its sheaf of denominators $\mathcal{I}_1$, i.e., the ideal sheaf in $\mathcal{O}_X$ whose sections over $V \in X_{\text{étale}}$ are

$$\mathcal{I}_1(V) := \{f \in \mathcal{O}_X(V) \mid fs \in \mathcal{L}(V)\};$$

compare [30, Definition 02P1]. Set $\mathcal{I}_2 := \mathcal{I}_1 \otimes \mathcal{L}^\vee$. Since the formation of the $\mathcal{I}_j$, $j = 1, 2$, is étale local, their properties may be reduced to the schematic case. Thus [30, Lemma 02P0] shows that the $\mathcal{I}_j$ are quasi-coherent sheaves of ideals and the corresponding closed subspaces $Y_j = V(\mathcal{I}_j)$ satisfy $\dim(Y_j) < \dim(X)$. By Lemma 1.2.2, induction applies so the $\mathcal{L}|_{Y_j}$ are ample. By construction, for each $m \geq 0$, there are exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_1 \otimes \mathcal{L}^{\otimes m} & \longrightarrow & \mathcal{L}^{\otimes m} & \longrightarrow & \mathcal{L}^{\otimes m}|_{Y_1} & \longrightarrow & 0 \\ & & \| & & & & & & \\ 0 & \longrightarrow & \mathcal{I}_2 \otimes \mathcal{L}^{\otimes(m-1)} & \longrightarrow & \mathcal{L}^{\otimes(m-1)} & \longrightarrow & \mathcal{L}^{\otimes(m-1)}|_{Y_2} & \longrightarrow & 0. \end{array}$$

Serre vanishing, [30, Lemma 0B5U], gives some $m_0 \geq 0$ such that, for all $m \geq m_0$, $H^i(Y_j, \mathcal{L}^{\otimes m}|_{Y_j}) = 0$ for all $i > 0$ and $j = 1, 2$. Thus comparing the long exact sequences in cohomology for the sequences above yields

$$\begin{aligned} h^i(X, \mathcal{L}^{\otimes m}) &= h^i(X, \mathcal{I}_1 \otimes \mathcal{L}^{\otimes m}) \\ &= h^i(X, \mathcal{I}_2 \otimes \mathcal{L}^{\otimes(m-1)}) = h^i(X, \mathcal{L}^{\otimes(m-1)}) \end{aligned}$$

for all $i \geq 2$ and $m \geq m_0$. Hence, for all $m \geq m_0$,

$$N := \sum_{i=2}^{\dim(X)} (-1)^i \, h^i(X, \mathcal{L}^{\otimes m})$$

is a constant. By definition of the intersection numbers, the leading coefficient of the numerical polynomial $\chi(X, \mathcal{L}^{\otimes m})$ is $(\mathcal{L}^{\dim X} \cdot X)$ and this is positive by assumption. Thus

$$\chi(X, \mathcal{L}^{\otimes m}) = h^0(X, \mathcal{L}^{\otimes m}) - h^1(X, \mathcal{L}^{\otimes m}) + N \to \infty \quad \text{as } m \to \infty.$$

So $h^0(X, \mathcal{L}^{\otimes m}) \to \infty$ and $\mathcal{L}^{\otimes m}$ is effective for $m \gg 0$. Ampleness is

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