New Handbook of Mathematical Psychology

Volume 2. Modeling and Measurement

The field of mathematical psychology began in the 1950s and includes both psychological theorizing, in which mathematics plays a key role, and applied mathematics motivated by substantive problems in psychology. Central to its success was the publication of the first *Handbook of Mathematical Psychology* in the 1960s. The psychological sciences have since expanded to include new areas of research, and significant advances have been made in both traditional psychological domains and in the applications of the computational sciences to psychology. Upholding the rigor of the original handbook, the *New Handbook of Mathematical Psychology* reflects the current state of the field by exploring the mathematical and computational foundations of new developments over the last half-century. The second volume focuses on areas of mathematics that are used in constructing models of cognitive phenomena and decision-making, and on the role of measurement in psychology.

WILLIAM H. BATCHELDER is Professor of Cognitive Sciences at the University of California, Irvine.

HANS COLONIUS is Professor of Psychology at Oldenburg University, Germany.

EHTIBAR N. DZHAFAROV is Professor of Psychological Sciences at Purdue University.

New Handbook of Mathematical Psychology

Volume 2. Modeling and Measurement

Edited by
William H. Batchelder
Hans Colonius
Ehtibar N. Dzhafarov

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314-321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India

79 Anson Road, #06-04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781009045407
DOI: 10.1017/9781139245906

First published 2018
First paperback edition 2021

A catalogue record for this publication is available from the British Library

Library of Congress Control Number: 2017303272

ISBN 978-1-107-02907-1 Hardback
ISBN 978-1-009-04540-7 Paperback

Contents

Contributors

F. GREGORY ASHBY, University of California, Santa Barbara (USA)

WILLIAM H. BATCHELDER, University of California, Irvine (USA)

JEROME R. BUSEMEYER, Indiana University, Bloomington (USA)

HANS COLONIUS, Oldenburg University (Germany)

ADELE DIEDERICH, Jacobs University Bremen (Germany)

CHRISTOPHER DOBLE, McGraw-Hill Education (USA)

JEAN-PAUL DOIGNON, Université Libre de Bruxelles (Belgium)

EHTIBAR N. DZHAFAROV, Purdue University (USA)

JEAN-CLAUDE FALMAGNE, University of California, Irvine (USA)

JOSEPH W. HOUPT, Wright State University (USA)

JÜRGEN HELLER, University of Tübingen (Germany)

BRETT JEFFERSON, Indiana University, Bloomington (USA)

MATT JONES, University of Colorado, Boulder (USA)

PETER D. KVAM, Michigan State University (USA)

KEIVAN MALLAHI-KARAI, Jacobs University Bremen (Germany)

LOUIS NARENS, University of California, Irvine (USA)

DONALD G. SAARI, University of California, Irvine (USA)

LUCA STEFANUTTI, University of Padova (Italy)

JAMES T. TOWNSEND, Indiana University, Bloomington (USA)

HAIZHANG ZHANG, Sun Yat-Sen University, Guangzhou (China)

JUN ZHANG, University of Michigan (USA)

Abbreviations

AHP, analytic hierarchy process
BAR, Borda assignment rule
BLIM, basic local independence model
CCN, computational cognitive neuroscience
cdf, cumulative distribution function
CRM, correct response model
DA, dopamine
DAT, DA active transporter
fMRI, functional magnetic resonance imaging
GLM, general linear model
Hrf, hemodynamic response function
IIA, independence of irrelevant alternatives
KL, Kullback–Leibler
KST, knowledge structure theory
LATER, linear approach to threshold with ergodic rate
LBA, linear ballistic accumulator
LCA, leaky competing accumulator
LCM, latent class model
LTD, long-term depression
LTP, long-term potentiation
MAP, maximum *a posteriori*
OU, Ornstein–Uhlenbeck
PCC, principle of correspondent change
pdf, probability (mass) distribution function
PFC, prefrontal cortex
POVM, positive operator value measurement
PSI, principle of selective influence
PSP, parameter space partitioning
QQ, quantum question
RKBS, reproducing kernel Banach space
RKHS, reproducing kernel Hilbert space
ROI, region of interest
RPE, reward prediction error
RSM, ratio scale matrix

RT, response time
SNpc, substantia nigra pars compacta
SPM, statistical parametric map
SPRT, sequential probability-ratio test
SRW, simple random walk
STDP, spike-timing-dependent plasticity
SVM, support vector machine
TMS, transcranial magnetic stimulation
VTA, ventral tegmental area

Preface

Volume 2 of the New Handbook of Mathematical Psychology (NHMP) continues our goal to emphasize mathematical foundations and mathematical themes in the psychological sciences rather than to emphasize empirical facts and specific competing models (see the preface to Volume 1). This second volume, subtitled Modeling and Measurement, focuses on areas of mathematics that are in major use in constructing formal models of cognitive phenomena as well as formal approaches to understanding the role of measurement in psychology.

The first five chapters in Volume 2 deal with probabilistic models for cognitive phenomena. In particular, the first four chapters show how the standard Kolmogorov measure-theoretic axioms of probability, random variables, and stochastic processes are employed to formalize cognitive models in a number of areas of psychology. In these chapters the standard axioms are presented along with derived concepts such as Markov processes (Chapters 1 and 2), martingales (Chapter 1), stochastic filtrations (Chapter 3), and the conditions needed to identify and test models (Chapter 4). Chapter 5 discusses a newer and increasingly popular approach to cognitive modeling by constructing probabilistic models using quantum probability axioms rather than the standard Kolmogorov axioms.

The final four chapters involve the foundational use of mathematics in areas of mathematical psychology that share connections to areas outside of psychology. Chapter 6 describes approaches that incorporate concepts from computational cognitive neuroscience; Chapter 7 takes up problems in voting theory, an area that overlaps with economics and political science; and Chapter 8 takes up problems in classification that arise in the area of machine learning in computer science. Finally, Chapter 9 presents the latest work on the concept of measure-theoretic meaningfulness, an area in foundations of measurement that overlaps with philosophy and the physical sciences. These four chapters mostly describe and use areas of mathematics that do not have a probabilistic character. In particular, concepts in geometry are used in Chapters 7 and 9, function spaces (Hilbert and Banach) are used in Chapter 8 (also see Chapter 5), and abstract algebra is seen in several of these chapters, especially Chapter 9.

Chapter 1 by Diederich and Mallahi-Karai discusses probabilistic models of decision-making. Such models are developed using the mathematics of stochastic processes, and the chapter provides a very complete coverage of a variety of stochastic processes that have been and are being used to model decision-making. Many of the models discussed involve discrete-time and continuous-time Markov processes; for example, discrete random walk models and continuous time diffusion processes such as the Wiener process and the Ornstein–Uhlenbeck process. The chapter shows how these diffusion processes can arise from limiting forms of the discrete models. Both uni- and multidimensional versions of these processes are formalized along with computational tools for dealing with them. In addition, other properties of stochastic processes such as stopping times and martingales are presented.

Chapter 2 by Jones takes up probabilistic models for binary decision-making, which are also discussed extensively in Chapter 1. Central in both of these chapters is the focus on probabilistic models that can account both for choice probabilities and the time it takes to make a response. Jones' chapter focuses on models that assume that the decision to choose one of the items in a presented pair is based on temporal accumulation of evidence samples. While the chapter discusses both discrete and continuous random walk models, a major effort is made to represent these models in terms of Bayesian inference theory. Particularly important is an extension of earlier work (e.g., Jones & Dzhafarov, 2014) that analyzes the effect of incorporating intertrial variability in the model parameters. This assumption is the main way for diffusion models to predict observable differences in processing times for correct and incorrect decisions. However, the models that employ intertrial variability have failed to provide a compelling rationale for their assumptions, and without a principled specification such models can predict any pattern of data. In other words, in the sense of falsifiability in Chapter 4, they cannot be rejected on data.

Chapter 3 by Houpt, Townsend, and Jefferson concerns properties of models for accounting for the time to complete a task consisting of multiple subcomponents, each of which must be processed to completion for the task itself to complete. Viewed generally, there are two general modeling schemes for such problems – serial processing and parallel processing of the components – and often models of both kinds can account for the same data (e.g., Townsend, 1972). The chapter provides a formal specification of serial and parallel models for such tasks cast in traditional Kolmogorov probability theory and the associated area of stochastic processes. These formalisms are presented along with the concept of filtration in stochastic processes. A key to the work discussed in the chapter is the way that selective influence is incorporated to compare models, namely where an environmental variable can selectively affect specific subprocesses in the task.

Chapter 4 by Doignon, Heller, and Stefanutti takes up important statistical issues concerning the scientific evaluation of parametric cognitive models for behavioral data. Such models have statistical parameters, and one important issue is whether a model is identified in the sense that different settings of the parameters necessarily generate different probability distributions of the observable data. Identified models allow unique measurement of their cognitive parameters based on observed data, and such models can be used to pinpoint the processes that are behind performance differences between participant groups. A related statistical issue is the question of whether a model is falsifiable (testable), that is, if it is possible, in principle, to observe data that a model cannot account for. Important work by Bamber and van Santen (2000) provided the formal definitions along with some mathematical tools to investigate these issues, and Chapter 4 greatly expands on these mathematical tools including the presentation of polytopes in multidimensional Euclidean space. The new tools are illustrated by their application to models from knowledge space theory (e.g., see Chapter 5 in the first volume; Doignon and Falmagne, 1999; or Falmagne and Doignon, 2011).

Chapter 5 by Busemeyer and Kvam provides mathematical tools and examples related to the probabilistic modeling of cognitive phenomena using formalisms from quantum theory. The past decade or so has seen an increasing use of principles from quantum theory in the specification of cognitive models. Busemeyer and co-workers have been major developers of this approach (e.g., Busemeyer, Wang, & Townsend, 2006). In order to explicate the formalisms of quantum probability, the authors present the necessary axioms of Hilbert spaces, leading up to linear operators, basis vectors, and tensor product spaces. Given this background, the quantum probability axioms are presented and contrasted with the classical probability theory. Then several examples are presented where cognitive models based on quantum formalisms provide successful explanations for cognitive phenomena that seem difficult to explain by more traditional cognitive models.

Chapter 6 by Ashby is about computational cognitive neuroscience (CCN). This area did not exist in the early days of mathematical psychology; however, since the 1990s recent advances in cognitive neuroscience have made this area attractive to modelers. Ashby was one of the first mathematical psychologists to embrace CCN models (e.g., Ashby *et al.,* 1998), and later he contributed to a mathematical understanding of fMRI data (e.g., Ashby, 2011). The chapter clearly shows how supplementing the usual behavioral data, e.g., response choices, response times, and confidence ratings, with detailed neurobiological data can greatly restrict a modeler's choices and consequently increase the scientific validity of a model. The chapter focuses on some important and well-developed models in neuroscience concerning such problems as single spiking neurons, models for firing rates, and models for learning such as Hebbian and reinforcement learning algorithms.

Chapter 7 by Saari concerns a problem in the area of decision-making, namely finding formal rules for aggregating the choices of a group of agents (voters, attributes, data sources) to generate a choice that best and most "fairly" represents the group. The problems raised in the chapter go back to the seminal work of Kenneth Arrow (e.g., 1951), where it was shown that seemingly reasonable axioms for aggregating voters' rank orders of options could not in general be satisfied. Following Arrow's work, a number of paradoxical examples have been given that appear to challenge the possibility of formalizing any satisfactory rule for fairly aggregating agents' choices. Saari has become a leading theorist in this area by laying bare the exact mathematical reasons behind the paradoxes in voting theory, (e.g., Saari, 1995) and this chapter focuses on new aggregation rules that involve derived or stated paired comparison choices. Unlike other chapters that develop probabilistic models of decision-making, Saari's chapter mostly uses algebraic and geometric methods to develop its results.

Chapter 8 by Zhang and Zhang discusses the problem of classification (categorization) at an advanced mathematical level involving Hilbert spaces and Banach spaces. The problem of classification is to induce a classifier from a finite set of classified instances (exemplars) that will perform well for other new instances. This is clearly an ill-posed problem because there are always an unlimited number of schemes that can perfectly classify a given finite set of exemplars. Many cognitive models have been developed to delineate classifiers that can predict data in categorization experiments. In the area of statistical machine learning, a number of approaches have been developed which are quite separate from the cognitive modeling work. A major approach in machine learning assumes that possible classifiers are elements in a Hilbert function space. The chapter develops related systems of classifiers by dropping the Hilbert space requirement of an inner product, and this leads the authors to develop related properties in a suitable Banach space. The relevant mathematics of these spaces along with the needed formalism to address the classification problem are presented in detail. Of special interest is that the standard cognitive models are also discussed, and their connection to specific versions of the machine learning models is explicated.

Chapter 9 by Falmagne, Narens, and Doble provides the latest foundational work on the concept of meaningfulness. Meaningfulness is an important subtopic of foundations of measurement (abstract measurement theory), and this area has been a major topic in mathematical psychology since its beginnings in the 1950s, (e.g. Krantz *et al.,* 1971; Narens, 1985; and Suppes & Zinnes, 1963). Meaningfulness concerns the logical status of propositions and lawful relationships involving measured quantities under permissible scale transformations of the quantities. Meaningfulness was first introduced in the context of psychophysics in a seminal article by Stevens (1946), and formal interest in meaningfulness was greatly stimulated by the article by Luce (1959), where he proposed that the measurement scales of variables in a psychophysical law placed formal restrictions on what functional forms (logarithmic, power function, etc.) are possible. Since Luce's article, the meaningfulness problem has been analyzed primarily by utilizing the concept of homomorphisms in abstract algebra and relating meaningfulness to dimensional analysis in physics and Klein's (1872) Erlanger program in geometry. The measure-theoretic concept of meaningfulness has been criticized by some authors (e.g., Guttman, 1971; Michell, 1986), especially in relation to dimensional analysis and the nature of dimensional constants in physics (Dzhafarov, 1995) The primary value of Chapter 9 for this volume, however, is in the mathematical themes it cogently introduces, and this value does not depend on possible views of the scientific status of its central concepts.

References

Arrow, K. J. (1951). *Social choice and individual values*. New York, NY: Wiley.
Ashby, F. G. (2011). *Statistical analysis of fMRI data*. Cambridge, MA: MIT Press.

Ashby, F. G., Alfonso-Reese, L. A., Turken, A. U., & Waldron, E. M. (1998). A neuropsychological theory of multiple systems in category learning. *Psychological Review*, *105*, 442–481.

Bamber, D., & van Santen, J. P. H. (2000). How to assess a model's testability and identifiability. *Journal of Mathematical Psychology*, *44*, 20–40.

Busemeyer, J. R., Wang, Z., & Townsend, J. (2006). Quantum dynamics of human decision making. *Journal of Mathematical Psychology*, *50*, 220–241.

Doignon, J.-P., & Falmagne, J. Cl. (1999). *Knowledge spaces*. Berlin: Springer-Verlag.

Doignon, J.-P., & Falmagne, J.-Cl. (2016). Knowledge spaces and learning spaces. In W. H. Batchelder, H. Colonius, E. N. Dzhafarov, & J. Myung (eds.), *New handbook of mathematical psychology* (pp. 274–321). Cambridge: Cambridge University Press.

Dzhafarov, E. N. (1995). Empirical meaningfulness, measurement-dependent constants, and dimensional analysis. In R. D. Luce, M. D'Zmura, D. Hoffman, G. J. Iverson, & A. K. Romney (eds.), *Geometric representations of perceptual phenomena* (pp. 113–134). Mahwah, NJ: Erlbaum.

Falmagne, J.-C., & Doignon, J.-P. (2011). *Learning spaces*. Berlin: Springer-Verlag.

Guttman, L. (1971). Measurement as structural theory. *Psychometrika*, *36*, 329–347.

Jones, M., & Dzhafarov, E. N. (2014). Unfalsifiability and mutual translatability of major modeling schemes for choice reaction time. *Psychological Review*, *121*, 1–32.

Klein, F. (1872). *Vergleichende Betrachtungen uber neuere geometrische Forschungen. Verlag on Andreas Deichert, Erlangen*. (Available online at the University of Michigan Historical Mathematics Collection.)

Krantz, D. H., Luce, R. D., Suppes, P., & Tversky, A. (1971). *Foundations of measurement, Volume 1: Additive and polynomial representations*. New York, NY: Academic Press.

Luce, R. D. (1959). On the possible psychophysical laws. *Psychological Review*, *66*, 81–95.

Michell, J. (1986). Measurement scales and statistics: A clash of paradigms. *Psychological Bulletin*, *100*, 398–407.

Narens, L. (1985). *Abstract measurement theory*. Cambridge, MA: MIT Press.

Saari, D. G. (1995). *Basic geometry of voting systems*. New York, NY: Springer-Verlag.

Stevens, S. S. (1946). On the theory of scales of measurement. *Science*, New Series, *103* (No. 2684), 677–680.

Suppes, P., & Zinnes, J. (1963). Basic measurement theory. In R. D. Luce, R. R. Bush, & E. Galanter (eds.), *Handbook of mathematical psychology*, *Vol. 1*. New York, NY: John Wiley & Sons.

Townsend, J. T. (1972). Some results concerning the identifiability of parallel and serial processes. *British Journal of Mathematical and Statistical Psychology*, *25*, 168–199.