The partition complex: an invitation to combinatorial commutative algebra

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Abstract

We provide a new foundation for combinatorial commutative algebra and Stanley-Reisner theory using the partition complex introduced in [1]. One of the main advantages is that it is entirely self-contained, using only a minimal knowledge of algebra and topology. On the other hand, we also develop new techniques and results using this approach. In particular, we provide

1. A novel, self-contained method of establishing Reisner's theorem and Schenzel's formula for Buchsbaum complexes.
2. A simple new way to establish Poincaré duality for face rings of manifolds, in much greater generality and precision than previous treatments.
3. A "master-theorem" to generalize several previous results concerning the Lefschetz theorem on subdivisions.
4. Proof for a conjecture of Kühnel concerning triangulated manifolds with boundary.

1 Introduction

Starting with the work of Hochster, Reisner and Stanley, powerful methods from commutative algebra developed by algebraic geometers could be used to provide a new and powerful way to study face numbers of simplicial and polyhedral complexes [9, 26].

However, using these powerful tools came with a drawback. First, they made the theory harder to access without background in commutative algebra. Second, even many of those applying them often used them as a black box, and the tools themselves became a distraction, leading to missed results and open questions that would otherwise have been simple.

And so, as a tourist might use an expensive lens to capture a vista, doing so suboptimally because he does not grasp its pros and cons, the physics of its makeup, we are left with pictures that feel somewhat lacking, blurry or hiding the important, leaving us dissatisfied.

So our goal here is twofold: To show how basic household means can take a much simpler, more gratifying picture, without sacrificing any of the generality. We then go a step further, and use the new methods to generalize the results with ease, using only the ingredients that can be found within the first algebra books you can find in your kitchen, and just a smidge of algebraic topology you find in every spice rack. As for combinatorics, we shall assume nothing beyond the most basic familiarity with simplicial complexes.

Hence, this is not so much a survey, as it is an attempt to build better and more powerful foundations, as well as offer newcomers a road towards research in the area, that is at the heart of new developments between combinatorics and Hodge Theory [18, 12, 2, 1]. Additionally, we offer also researchers in combinatorial commutative algebra a more consistent and stronger set of tools. We are therefore a little curt on direct combinatorial applications, for which we refer to the initial sections of [1], and instead offer an focused introduction to the techniques.

1.1 Overview

Before we begin discussing the details, let us provide a little motivation. We want to understand various combinatorial invariants of simplicial complexes. Most basic among these is the face vector, counting the number of vertices, edges, and so on. We may wish to restrict the class of complexes under investigation: for example, to look only at planar graphs, or at simplicial complexes that triangulate a surface. The restrictions we place are usually homological in nature.

The issue is then how the combinatorics and topology come together. The trick is to use rings which contain information from both worlds.

Indeed, one of the key observations of combinatorial commutative algebra was the realization that the homological properties of a simplicial complex are encoded in its so called *face ring* in a variety of ways, often first glimpsed and disseminated as unpublished ideas and results of Hochster[1]. The first key result here is Reisner's theorem (discussed in Section 4), that connects the vanishing of homology over a fixed field to the Cohen-Macaulay property of the associated face ring. Here, not only the global homology of the simplicial complex comes into play, but also the homology of principal filters in the face lattice.

The essentially only proof available for this theorem goes via the local cohomology as introduced by Grothendieck in the 1960s, and most of the following research has similarly employed the same tool. We instead use the partition complex here, a significantly more down-to-earth tool that has several direct benefits, most of all that one can see what happens in a surgical way.

We also obtain the generalization to manifolds, due to Schenzel [22], which is relatively transparent at least to experts, but has the drawback that it is, in parts, only available in his German thesis. We provide this in Section 6.

Our next stop in the way is a new way to address and understand a fundamental property of intersection rings that arises in the context of combinatorial Hodge theory: Poincaré duality. Again we offer a new transparent proof of Poincaré duality for the face rings of spheres, and then proceed to provide generalizations to arbitrary manifolds, discussed in Section 7.

Finally, we discuss some new applications to face number problems for manifolds. In Section 9, we discuss the connection to subdivisions and Lefschetz properties, and provide a far-reaching subdivision theorem, providing a common generalization of previous works in one swoop. We also discuss related conjectures of Kühnel, concerning small triangulations of manifolds.

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[1]This seems to justify the old adage that discoveries are never named after their discoverer, for the other name of face rings, Stanley-Reisner rings, makes no mention of Hochster

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2 Preliminaries

In this section we set up some basic notation and definitions. Experienced readers can skip most of the text, but may still wish to look at the notation and at definitions for relative simplicial complexes, as well as the corresponding modules over face rings.

2.1 Simplicial complexes and face rings

2.1.1 Simplicial complexes We begin by recalling some common definitions.

Definition A simplicial complex Δ is a downwards-closed family of subsets of a finite set called the ground set. The ground set is usually left implicit or taken to be $[n] = \{1, \ldots, n\}$ for some n. Being downwards-closed means that if $\tau \in \Delta$ and $\rho \subset \tau$ then $\rho \in \Delta$.

In particular, if a simplicial complex is nonempty, it contains $\emptyset$ as a face. Thus the complex $\{\emptyset\}$ contains no nonempty faces, but is different than the void complex $\emptyset$.

A subcomplex of a simplicial complex is a subset which is itself a simplicial complex.

An element $\tau \in \Delta$ is called a simplex or a face. Its dimension is $\dim(\tau) = |\tau| - 1$, and the dimension of Δ is $\max_{\tau\in\Delta} \dim(\tau)$. A face of Δ is called a facet if its dimension equals $\dim(\Delta)$. Faces of dimension zero and one are called vertices and edges respectively.

Definition Let Δ be a simplicial complex. The *star* of a simplex τ is the simplicial complex $\mathrm{st}_\tau(\Delta) = \{\rho \in \Delta \mid \tau \cup \rho \in \Delta\}$. The *link* of τ is $\mathrm{lk}_\tau(\Delta) = \{\rho \in \Delta \mid \tau \cup \rho \in \Delta, \tau \cap \rho = \emptyset\}$.

The k-faces of Δ are denoted by $\Delta^{(k)} = \{\tau \in \Delta \mid \dim(\tau) = k\}$, and the k-skeleton $\Delta^{(\leq k)}$ is the subcomplex consisting of faces of dimension at most k.

In one or two places we use the join and subtraction operations. For simplicial complexes Δ_1, Δ_2 on disjoint ground sets, the join is $\Delta_1^* \Delta_2 = \{\tau \cup \rho \mid \tau \in \Delta_1, \rho \in \Delta_2\}$, a simplicial complex on the union of the ground sets of Δ_1 and Δ_2.

If Δ is a simplicial complex and τ is a face, $\Delta - \tau$ is the maximal subcomplex which does not contain τ. Its faces are $\{\sigma \in \Delta \mid \sigma \cap \tau = \emptyset\}$.

It is worth noting that simplicial complexes are not equivalent to semi-simplicial sets (sometimes called Δ-complexes by Hatcher).

2.1.2 Relative simplicial complexes We work with relative simplicial complexes analogously to how one often works with pairs of topological spaces. The theory generalizes smoothly to this setting, which is sometimes cleaner. See also [3, 1].

Definition A relative simplicial complex $\Psi = (\Delta, \Gamma)$ is a pair consisting of a simplicial complex Δ and a subcomplex Γ. Its faces are $\Delta \setminus \Gamma$, i.e. the non-faces of Γ. In particular, $\dim(\Psi) = \max_{\tau \in \Psi} \dim(\tau)$ can be smaller than $\dim(\Delta)$, and it is possible for $\emptyset$ not to be a face. Any simplicial complex Δ can be treated in this language as the relative complex $(\Delta, \emptyset)$.

The star of a simplex τ within Ψ is $\mathrm{st}_\tau \Psi = (\mathrm{st}_\tau \Delta, \mathrm{st}_\tau \Gamma)$. Similarly, the link is $\mathrm{lk}_\tau \Psi = (\mathrm{lk}_\tau \Delta, \mathrm{lk}_\tau \Gamma)$.

A relative complex Ψ is pure if all its maximal faces have the same dimension.

Many basic lemmas about simplicial complexes work for relative complexes as well, and we will often extend definitions from absolute to relative without further mention. For instance, if $\Psi = (\Delta, \Gamma)$ and $\tau \in \Delta$ then $\mathrm{st}_\tau \Psi = \tau^* \mathrm{lk}_\tau \Psi = (\tau^* \mathrm{lk}_\tau \Delta, \tau^* \mathrm{lk}_\tau \Gamma)$. Note that the join of any complex with the void complex is void.

The open star of a face τ in a simplicial complex is usually defined to be the set of faces containing τ. This is not a subcomplex in the usual sense, but we can define a relative complex to fill the same role.

Definition Let Δ be a simplicial complex. The open star of a face τ is $\mathrm{st}^\circ_\tau \Delta = (\mathrm{st}_\tau \Delta, \mathrm{st}_\tau \Delta - \tau)$.

2.1.3 Homology of complexes The cohomology $H^*(\Psi; \Bbbk)$ of a relative complex $\Psi = (\Delta, \Gamma)$ is the simplicial cohomology of the pair with coefficients in $\Bbbk$. For a complex Δ, we consider $\emptyset \in \Delta$ as a face (of dimension -1) for this purpose. Thus our $H^*(\Delta)$ is what is often denoted $\tilde{H}^*(\Delta)$. In particular the void complex $\emptyset$ has vanishing cohomology in all dimensions, but $\Delta = \{\emptyset\}$ has

$$H^i(\Delta; \Bbbk) = \begin{cases} \Bbbk & i = -1 \\ 0 & \text{otherwise.} \end{cases}$$

2.1.4 Face rings Face rings, or Stanley-Reisner rings, are main object of the paper. Our treatment is standard except for the relative case, in which we follow [3] and [1].

Fix a field $\Bbbk$. Except in Section 3, $\Bbbk$ is assumed to be infinite. This is a harmless assumption, as field extensions change no property that interests us in this context.

Definition Let Δ be a simplicial complex. Define the polynomial ring $\Bbbk[x_v \mid v \in \Delta^{(0)}]$, with variables indexed by vertices of Δ. The Stanley-Reisner ideal (or non-face ideal) I_Δ of Δ is the ideal generated by all elements of the form $x_{v_1} \cdot x_{v_2} \cdot \ldots \cdot x_{v_j}$ where $\{v_1, \ldots, v_j\}$ is not a face of Δ.

The Stanley-Reisner ring (or face ring) of Δ is

$$\Bbbk[\Delta] := \Bbbk[x_v \mid v \in \Delta^{(0)}]/I_\Delta.$$

If $\Psi = (\Delta, \Gamma)$ is a relative complex, the *relative face module* of Ψ is defined by I_Γ / I_Δ. This is an ideal of $\Bbbk[\Delta]$.

Two main types of maps between face rings and modules are used in this paper. If $\Psi = (\Delta, \Gamma)$ and $\Psi' = (\Delta, \Gamma')$ are relative complexes such that $\Gamma' \subset \Gamma$, there is an inclusion map

$$\mathbb{k}[\Psi] \hookrightarrow \mathbb{k}[\Psi'].$$

Similarly, if $\Psi = (\Delta, \Gamma)$ and $\Psi' = (\Delta', \Gamma)$ such that $\Delta' \subset \Delta$ is a subcomplex, there is a restriction map

$$\mathbb{k}[\Psi] \twoheadrightarrow \mathbb{k}[\Psi'].$$

In general, maps do not exist in the opposite direction. Two particularly relevant examples are the inclusion of an open star into a complex and the restriction to the star of a face. Explicitly, for $\Psi = (\Delta, \Gamma)$ and any $\tau \in \Delta$, these are maps

$$\mathbb{k}[\mathrm{st}^\circ_\tau \Psi] \simeq \mathbb{k}[\Delta, \mathrm{st}_\tau \Gamma \cup (\Delta - \tau)] \to \mathbb{k}[\Psi]$$

and

$$\mathbb{k}[\Psi] \to \mathbb{k}[\mathrm{st}_\tau \Psi]$$

respectively.

2.1.5 Gradings of face rings Face rings can be *graded* by monomial degree. That is, if Δ is a complex, we can write

$$\mathbb{k}[\Delta] = \bigoplus_{n \geq 0} \mathbb{k}[\Delta]_n,$$

where the direct sum is a sum of vector spaces over $\mathbb{k}$, and $\mathbb{k}[\Delta]_n$ is the subspace spanned by monomials of degree n. This is called the coarse grading. An element of $\mathbb{k}[\Delta]$ is homogeneous if it is in a single graded piece, or in other words, if it is a linear combination of monomials having the same degree.

There is also a finer grading, by the exponent vectors of monomials. If Δ has vertices $\{v_1, \ldots, v_k\}$ and $x^\alpha = x_{v_1}^{\alpha_1} \cdot \ldots \cdot x_{v_k}^{\alpha_k}$ is a monomial, its exponent vector is $(\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_{\geq 0}^{\Delta^{(0)}}$. The piece of $\mathbb{k}[\Delta]$ in degree α is the span of this monomial. This is the fine grading.

Note that in both cases, the degree of a product of homogeneous elements is the sum of their degrees.

Given a graded module or algebra, one can encode the dimensions of the graded pieces in a generating function. This is called the *Hilbert series*, and we shall focus mostly on the Hilbert series of a face ring with respect to the coarse grading

$$H(\mathbb{k}[\Delta])(t) \;= \sum_{i=0}^{\infty} \dim_{\mathbb{k}}((\mathbb{k}_i[\Delta])) \cdot t^i.$$

MotivationThis interests us because the Hilbert series of a simplicial complex is also combinatorial:

$$H(\mathbb{k}[\Delta])(t) \;=\; \frac{1}{(1-t)^n} \sum_{i=0}^{d} f_{i-1} t^i (1-t)^{n-i}.$$

The same discussion applies verbatim to relative face modules. The relevance of this is that maps between modules often preserve the degree. In this case, we can often understand a complex of maps most easily by examining each graded piece separately.

2.2 Chain complexes

We discuss some definitions and basic lemmas for chain and double complexes, and provide a basic introduction. If you have not seen chain complexes before, we recommend Hatcher for a basic introduction [8].

Definition [Chain complexes and tensor products] All our complexes are cohomologically graded. That is, our chain complexes are denoted C^*, with differential $C^* \to C^{*+1}$. It is convenient to call $H^i(C^*)$ the i-th homology, rather than cohomology, of C^*.

To shift the index by p, we write C^{*+p} (and $(C^{*+p})^i = C^{i+p}$).

If (B^*, d) and (C^*, d') are chain complexes, their tensor product is the double complex $T^{*,*}$ defined by

$$T^{i,j} = B^i \otimes C^j$$

together with maps $d^h = d \otimes \mathrm{id} : T^{i,j} \to T^{i+1,j}$ and $d^v = \mathrm{id} \otimes d' : T^{i,j} \to T^{i,j+1}$. If B^*, C^* are complexes of modules over some ring R, the tensor product is of R modules, i.e. it is $B^i \otimes_R C^j$. Note the convention here is that the squares of the complex commute.

A small piece of $T^{*,*}$ can be pictured as follows.

$$\begin{tikzcd}
 & \cdots & \cdots & \\
\cdots \arrow[r, "d^h"] & B^i \otimes C^{j+1} \arrow[u, "d^v"] \arrow[r, "d^h"] & B^{i+1} \otimes C^{j+1} \arrow[u, "d^v"] \arrow[r, "d^h"] & \cdots \\
\cdots \arrow[r, "d^h"] & B^i \otimes C^j \arrow[u, "d^v"] \arrow[r, "d^h"] & B^{i+1} \otimes C^j \arrow[u, "d^v"] \arrow[r, "d^h"] & \cdots \\
 & \cdots \arrow[u, "d^v"] & \cdots \arrow[u, "d^v"] & \cdots
\end{tikzcd}$$

2.3 Double complexes

The main proofs of the paper are established using the homology of double complexes, the homological way to perform what combinatorialists know well as double counting. To do this in a manner as accessible as possible, without leaving too much for the reader, we use mapping cones very extensively.

Everything we need is introduced below.

We begin with some notation.

Definition Let $C^{*,*}$ be a double complex with commuting differentials d^h and d^v (our double complexes always have commuting differentials). Each row $C^{*,j}$ and each column $C^{i,*}$ is a chain complex with differential induced from d^h or d^v respectively.

The *total complex* of C is a chain complex given by $\mathrm{Tot}(C)^k = \bigoplus_{i+j=k} C^{i,j}$ and differential $d^k_{\mathrm{Tot}} : \mathrm{Tot}(C)^k \to \mathrm{Tot}(C)^{k+1}$ defined by either $d^h + (-1)^k d^v$ or $d^v + (-1)^k d^h$. These give equivalent homology, and it is convenient to have both (an alternative is to transpose the complex, but both are used for the same double complex here).

We denote elements $\alpha \in \mathrm{Tot}(C)^k$ by sums $\alpha = \sum_{i+j=k} \alpha^{i,j}$, where it is understood that $\alpha^{i,j} \in C^{i,j}$.

The *truncation* $C^{*\geq i_0,*}$ is a double complex defined by

$$(C^{*\geq i_0,*})^{i,j} = C^{i\geq i_0,j} = \begin{cases} C^{i,j} & i \geq i_0 \\ 0 & \text{otherwise,} \end{cases}$$

with the same differentials as $C^{*,*}$, and 0 for $i < i_0$.

The truncation $C^{*,*\leq j_1}$ is defined analogously.

Our goal for the rest of this section is to produce exact sequences tying together the rows, columns, and total complex of a double complex. We do this using mapping cones. The idea is introduced after a little preparation.

Lemma 2.1 *Let $C^{*,*}$ be a bounded double complex. For $H^k(\mathrm{Tot}(C))$ to vanish, it suffices that the homology in the vertical direction of $C^{i,k-i}$ is zero for each i, i.e. that $H^{k-i}(C^{i,*}) = 0$ for all i. Similarly, it suffices that $H^{k-i}(C^{*,i}) = 0$ for all i.*

Proof We show this for the vertical case, the horizontal one being analogous. Let $\sum_{i+j=k} \alpha^{i,j} \in \mathrm{Tot}(C)^k$ be a cycle, and let i_0 be the minimial index such that $\alpha^{i_0,k-i_0} \neq 0$. Then $d^v(\alpha^{i_0,k-i_0}) = 0$, so by assumption there is some $\beta = \beta^{i_0,k-i_0-1}$ mapping to α^{i_0,k_0} under d^v.

$$\begin{tikzcd}
\ddots \arrow[r] & {} \\
& 0 \arrow[u] \arrow[r] & 0 \\
& & \alpha^{i_0,k-i_0} \arrow[u] \arrow[r] & 0 \\
& & \beta \arrow[u] \arrow[r] & \alpha^{i_0+1,k-i_0-1} \arrow[u] \arrow[r] & 0 \\
& & & & \ddots \arrow[u]
\end{tikzcd}$$

Thus $\alpha' = \alpha - ((-1)^{k-1}d^h + d^v)(\beta)$ differs from α by a boundary. Replacing α by α' increases the minimal nonvanishing index i_0, and after finitely many steps the process terminates because $C^{*,*}$ is bounded. □

Corollary 2.2 *If all rows or all columns of a double complex are exact then the total complex is acyclic.*

We introduce maps $\mathfrak{R}, \mathfrak{U}$ (for "right" and "up") between columns (respectively rows) of a double complex and the total complexes of certain truncations.

Definition Let $C^{*,*}$ be a double complex. There is a chain map

$$\mathfrak{R}^i : C^{i,*} \to \mathrm{Tot}(C^{*\geq i+1,*})^{*+i+1},$$

from the i-th column to the total complex of a truncation of $C^{*,*}$, which is given by

$$\begin{aligned} C^{i,j} &\to \mathrm{Tot}(C^{*\geq i+1,*})^{i+j+1} \\ \alpha &\mapsto d^h(\alpha) \in C^{i+1,j} \subset \bigoplus_{\substack{r+s=i+j+1,\\ r\geq i+1}} C^{r,s}. \end{aligned}$$

For the signs make $\mathfrak{R}$ commute with the differentials, the differential of the total complex is taken to be $d^v + (-1)^k d^h$.

This is illustrated below, with summands of $\mathrm{Tot}(C^{*\geq i+1,*})^{i+j+1}$ underlined.

$$\begin{tikzcd}
{} & & & \\
C^{i,j+1} \arrow[u] & {} & & \\
C^{i,j} \arrow[u] \arrow[r] & \underline{C^{i+1,j}} \arrow[u] \arrow[r] & {} & \\
C^{i,j-1} \arrow[u] & & \underline{C^{i+2,j-1}} \arrow[u] \arrow[r] & {} \\
\vdots & & & \ddots
\end{tikzcd}$$

There is a similar chain map

$$\mathfrak{L}^j : \mathrm{Tot}(C^{*,*\leq j})^* \to C^{*-j,j+1},$$

from the total complex of a truncation of $C^{*,*}$ to the $j+1$-th row. On an element

$$\alpha = \sum_{r+s=k} \alpha^{r,s} \in \mathrm{Tot}(C^{*,*\leq j})^k$$

we define it by

$$\mathfrak{L}^j(\alpha) = d^v(\alpha^{k-j,j}).$$

Here the differential of the total complex should be taken to be $d^h + (-1)^k d^v$.

Definition [Mapping cones] Let $f : (C^*, \partial) \to (C'^*, \partial')$ be a map of chain complexes. The mapping cone of f is the chain complex $(M(f)^*, d)$, where $M(f)^i = C^i \oplus C'^{i-1}$ and $d^i(\alpha, \beta) = (\partial\alpha, \partial'\beta + (-1)^i f\alpha)$.

Given f, we can construct a map of chain complexes $\iota : C'^{*-1} \to M(f)^*$ by $\beta \mapsto (0, \beta)$. This fits into a short exact sequence

$$0 \to C'^{*-1} \xrightarrow{\iota} M(f)^* \to C^* \to 0,$$

which gives rise to a long exact sequence

$$\ldots \to H^i(C) \to H^i(C') \to H^{i+1}(M(f)) \to H^{i+1}(C) \to \ldots$$

in which the connecting homomorphism is induced by f.

The next lemma is the essential point.

Lemma 2.3 *Let $C^{*,*}$ be a double complex with commuting vertical and horizontal maps d^h, d^v. There are isomorphisms*

$$M(\mathfrak{R}^i) \simeq \mathrm{Tot}(C^{*\geq i,*})^{*+i}$$
$$M(\mathfrak{L}^j) \simeq \mathrm{Tot}(C^{*,*\leq j+1})^{*}.$$

Proof First consider $f = \mathfrak{R}^i : C^{i,*} \to \mathrm{Tot}(C^{*\geq i+1,*})^{*+i+1}$. By definition,

$$M(f)^j = C^{i,j} \oplus \mathrm{Tot}(C^{*\geq i+1,*})^{i+j+1-1} = \bigoplus_{\substack{r+s=i+j \\ r\geq i}} C^{r,s} = \mathrm{Tot}(C^{*\geq i,*})^{i+j},$$

and the differential of $M(f)^*$ is essentially the same as that of $\mathrm{Tot}(C^{*\geq i,*})$.

Now consider $f = \mathfrak{L}^j : \mathrm{Tot}(C^{*,*\leq j})^* \to C^{*-j,j+1}$. This time

$$M(f)^i = \mathrm{Tot}(C^{*,*\leq j})^i \oplus C^{i-1-j,j+1} = \bigoplus_{\substack{r+s=i \\ s\leq j+1}} C^{r,s} = \mathrm{Tot}(C^{*,*\leq j+1})^i,$$

and the differential of $M(f)^*$ is the same as that of the total complex on the right hand side above if j is even. If j is odd, it is harmless to modify the differential of the mapping cone to be

$$d^i(\alpha, \beta) = (\partial\alpha, \partial'\beta + (-1)^{i-1} f\alpha)$$

instead of the expression above: the two expressions give isomorphic complexes $M(f)^*$. □

Remark Mapping cones are a construction in homological algebra, motivated by a similar construction in algebraic topology. They are found in most textbooks on homological algebra, sometimes with slightly different indexing or sign conventions. The topological construction from which they originate is described, for instance, in chapter 0 of Hatcher's text [8].

3 Cohen-Macaulay Complexes and why we care

Let us now turn to the little bit of commutative algebra necessary for our purposes. We refer to [5] for a general account, and [6] for something a little more specialized to our situation.

3.1 The Basic Idea

Consider a simplicial complex Δ and its face ring $\Bbbk[\Delta]$: if Δ has at least one vertex v, this is a graded ring with $\Bbbk[\Delta]_i \neq 0$ for each i. Indeed $x_v^i \in \Bbbk[\Delta]_i$. Thus, as a vector space over $\Bbbk$, each graded piece has finite dimension, but the entire ring is always infinite dimensional. It is useful to work with a finite-dimensional $\Bbbk$-algebra instead, provided it preserves enough information about $\Bbbk[\Delta]$. The idea then is to "peel" $\Bbbk[\Delta]$ by quotienting out an ideal which is as large as reasonably possible.

That $\Bbbk[\Delta]$ is Cohen-Macaulay means this peeling can be performed especially nicely, as we shall soon see.

The importance of all this is due to the fact that $\Bbbk[\Delta]$ is always Cohen-Macaulay if Δ is the link or star of a face in any triangulation of a manifold with boundary (and in particular if Δ is a disk or a sphere). This is a shadow of the fact that each point of a manifold has a neighborhood with trivial topology.

We put rings and modules on an equal footing, so these tools are later available for relative face modules.

Definition Let R be a ring and M an R-module. A *regular sequence* on M, or *M-sequence*, is a sequence of elements $(\theta_1, \dots, \theta_n)$ in R such that:

1. Each θ_i is a nonzerodivisor on $M/\langle\theta_1, \dots, \theta_{i-1}\rangle$, and
2. $M/\langle\theta_1, \dots, \theta_n\rangle \neq 0$.

If R is a graded ring, a sequence as above is called homogeneous if each θ_i is.

We care mainly about homogeneous regular sequences, and among them mainly about those in which all elements have degree 1. We will see that if the field $\Bbbk$ is infinite, A and M are graded with A generated in degree 1 (in particular $A_0 = \Bbbk$), and there exists an M-sequence of length n, then there also exists an M-sequence consisting of degree-1 elements.

Quotienting a graded $\Bbbk$-algebra A by a regular sequence of degree-1 elements is an operation which is well-behaved with respect to the *Hilbert series* of A, which we previously encountered in the case $A = \Bbbk[\Delta]$. For a graded $\Bbbk$-vector space V, the Hilbert series is

$$H_V(t) = \sum_{i=0}^{\infty} \dim_\Bbbk(V_i) \cdot t^i.$$

Consider the ideal $\langle\theta_1\rangle$ generated by a nonzerodivisor of A having degree 1. Since the multiplication map by θ_1 is an injection of vector spaces $A \to A$ which increases the degree by 1, we have

$$\dim_\Bbbk(\langle\theta_1\rangle_i) = \dim_\Bbbk(A_{i-1}),$$

so

$$H_{\langle\theta_1\rangle}(t) = \sum_{i=0}^{\infty} \dim_\Bbbk(A_{i-1}) \cdot t^i = t \cdot H_{A(t).}$$

In particular, we find that

$$H_{A/\langle\theta_1\rangle}(t) = H_A(t) - H_{\langle\theta_1\rangle}(t) = (1-t) \cdot H_A(t).$$

Modding out by the ideal $\langle\Theta\rangle$ generated by a regular sequence $\Theta = (\theta_1, \dots, \theta_n)$ consisting of degree-1 elements therefore gives

$$H_{A/\langle\Theta\rangle}(t) = (1-t)^n \cdot H_A(t).$$

All this works in just the same way if we instead work with a regular sequence of degree-1 elements on an A-module M.