

Contents

Preface	xv
Introduction	1
Conventions and notation	3
1 Definitions and Basic Properties	6
a Definition	6
b Basic properties of algebraic groups	12
c Algebraic subgroups	18
d Examples	22
e Kernels and exact sequences	23
f Group actions	26
g The homomorphism theorem for smooth groups	28
h Closed subfunctors: definitions and statements	29
i Transporters	30
j Normalizers	31
k Centralizers	33
l Closed subfunctors: proofs	35
Exercises	38
2 Examples and Basic Constructions	39
a Affine algebraic groups	39
b Étale group schemes	44
c Anti-affine algebraic groups	45
d Homomorphisms of algebraic groups	46
e Products	49
f Semidirect products	50
g The group of connected components	51
h The algebraic subgroup generated by a map	53
i Restriction of scalars	56
j Torsors	60
Exercises	61
3 Affine Algebraic Groups and Hopf Algebras	64
a The comultiplication map	64
b Hopf algebras	65

c	Hopf algebras and algebraic groups	66
d	Hopf subalgebras	67
e	Hopf subalgebras of $\mathcal{O}(G)$ versus subgroups of G	68
f	Subgroups of $G(k)$ versus algebraic subgroups of G	68
g	Affine algebraic groups in characteristic zero are smooth	69
h	Smoothness in characteristic $p \neq 0$	72
i	Faithful flatness for Hopf algebras	73
j	The homomorphism theorem for affine algebraic groups	74
k	Forms of algebraic groups	76
	Exercises	81
4	Linear Representations of Algebraic Groups	83
a	Representations and comodules	83
b	Stabilizers	85
c	Representations are unions of finite-dimensional representations	86
d	Affine algebraic groups are linear	86
e	Constructing all finite-dimensional representations	88
f	Semisimple representations	90
g	Characters and eigenspaces	92
h	Chevalley's theorem	94
i	The subspace fixed by a group	96
	Exercises	97
5	Group Theory; the Isomorphism Theorems	98
a	The isomorphism theorems for abstract groups	98
b	Quotient maps	99
c	Existence of quotients	102
d	Monomorphisms of algebraic groups	106
e	The homomorphism theorem	108
f	The isomorphism theorem	111
g	The correspondence theorem	112
h	The connected-étale exact sequence	114
i	The category of commutative algebraic groups	115
j	Sheaves	116
k	The isomorphism theorems for functors to groups	118
l	The isomorphism theorems for sheaves of groups	118
m	The isomorphism theorems for algebraic groups	119
n	Some category theory	121
	Exercises	121
6	Subnormal Series; Solvable and Nilpotent Algebraic Groups	124
a	Subnormal series	124
b	Isogenies	126
c	Composition series for algebraic groups	127
d	Derived groups and commutator groups	129

Contents

ix

e	Solvable algebraic groups	131
f	Nilpotent algebraic groups	133
g	Existence of a largest algebraic subgroup with a given property	134
h	Semisimple and reductive groups	135
i	A standard example	136
7	Algebraic Groups Acting on Schemes	138
a	Group actions	138
b	The fixed-point subscheme	138
c	Orbits and isotropy groups	139
d	The functor defined by projective space	141
e	Quotients of affine algebraic groups	141
f	Linear actions on schemes	145
g	Flag varieties	146
	Exercises	146
8	The Structure of General Algebraic Groups	148
a	Summary	148
b	Normal affine algebraic subgroups	149
c	Pseudo-abelian varieties	149
d	Local actions	150
e	Anti-affine algebraic groups and abelian varieties	151
f	Rosenlicht's decomposition theorem.	152
g	Rosenlicht's dichotomy	153
h	The Barsotti–Chevalley theorem	154
i	Anti-affine groups	156
j	Extensions of abelian varieties by affine algebraic groups	159
k	Homogeneous spaces are quasi-projective	160
	Exercises	162
9	Tannaka Duality; Jordan Decompositions	163
a	Recovering a group from its representations	163
b	Jordan decompositions	166
c	Characterizing categories of representations	172
d	Categories of comodules over a coalgebra	174
e	Proof of Theorem 9.24	178
f	Tannakian categories	183
g	Properties of G versus those of $\text{Rep}(G)$	184
10	The Lie Algebra of an Algebraic Group	186
a	Definition	186
b	The Lie algebra of an algebraic group	188
c	Basic properties of the Lie algebra	190
d	The adjoint representation; definition of the bracket	191
e	Description of the Lie algebra in terms of derivations	194

f	Stabilizers	196
g	Centres	197
h	Centralizers	197
i	An example of Chevalley	198
j	The universal enveloping algebra	199
k	The universal enveloping p -algebra	205
l	The algebra of distributions (hyperalgebra) of an algebraic group	207
	Exercises	208
11	Finite Group Schemes	209
a	Generalities	209
b	Locally free finite group schemes over a base ring	211
c	Cartier duality	212
d	Finite group schemes of order p	214
e	Derivations of Hopf algebras	215
f	Structure of the underlying scheme of a finite group scheme	218
g	Finite group schemes of order n are killed by n	220
h	Finite group schemes of height at most one	221
i	The Verschiebung morphism	224
j	The Witt schemes W_n	226
k	Commutative group schemes over a perfect field	227
	Exercises	229
12	Groups of Multiplicative Type; Linearly Reductive Groups	230
a	The characters of an algebraic group	230
b	The algebraic group $D(M)$	230
c	Diagonalizable groups	233
d	Diagonalizable representations	234
e	Tori	236
f	Groups of multiplicative type	236
g	Classification of groups of multiplicative type	239
h	Representations of a group of multiplicative type	241
i	Density and rigidity	242
j	Central tori as almost-factors	245
k	Maps to tori	247
l	Linearly reductive groups	248
m	Unirationality	250
	Exercises	252
13	Tori Acting on Schemes	254
a	The smoothness of the fixed-point subscheme	254
b	Limits in schemes	258
c	The concentrator scheme in the affine case	260
d	Limits in algebraic groups	264
e	Luna maps	267

Contents

xi

f	The Białynicki-Birula decomposition	272
g	Proof of the Białynicki-Birula decomposition	277
	Exercises	278
14	Unipotent Algebraic Groups	279
a	Preliminaries from linear algebra	279
b	Unipotent algebraic groups	280
c	Unipotent elements in algebraic groups	286
d	Unipotent algebraic groups in characteristic zero	288
e	Unipotent algebraic groups in nonzero characteristic	292
f	Algebraic groups isomorphic to \mathbb{G}_a	298
g	Split and wound unipotent groups	299
	Exercises	301
15	Cohomology and Extensions	302
a	Crossed homomorphisms	302
b	Hochschild cohomology	304
c	Hochschild extensions	307
d	The cohomology of linear representations	310
e	Linearly reductive groups	311
f	Applications to homomorphisms	312
g	Applications to centralizers	313
h	Calculation of some extensions	313
	Exercises	323
16	The Structure of Solvable Algebraic Groups	324
a	Trigonalizable algebraic groups	324
b	Commutative algebraic groups	327
c	Structure of trigonalizable algebraic groups	330
d	Solvable algebraic groups	335
e	Connectedness	338
f	Nilpotent algebraic groups	340
g	Split solvable groups	343
h	Complements on unipotent algebraic groups	346
i	Tori acting on algebraic groups	347
	Exercises	351
17	Borel Subgroups and Applications	352
a	The Borel fixed-point theorem	352
b	Borel subgroups and maximal tori	354
c	The density theorem	361
d	Centralizers of tori	363
e	The normalizer of a Borel subgroup	366
f	The variety of Borel subgroups	368
g	Chevalley's description of the unipotent radical	370

h	Proof of Chevalley's theorem	373
i	Borel and parabolic subgroups over an arbitrary base field	375
j	Maximal tori and Cartan subgroups over an arbitrary field	376
k	Algebraic groups over finite fields	383
l	Split algebraic groups	384
	Exercises	385
18	The Geometry of Algebraic Groups	387
a	Central and multiplicative isogenies	387
b	The universal covering	388
c	Line bundles and characters	390
d	Existence of a universal covering	392
e	Applications	394
f	Proof of theorem 18.15	395
	Exercises	396
19	Semisimple and Reductive Groups	397
a	Semisimple groups	397
b	Reductive groups	399
c	The rank of a group variety	402
d	Deconstructing reductive groups	403
	Exercises	406
20	Algebraic Groups of Semisimple Rank One	407
a	Group varieties of semisimple rank 0	407
b	Homogeneous curves	408
c	The automorphism group of the projective line	409
d	A fixed-point theorem for actions of tori	410
e	Group varieties of semisimple rank 1.	411
f	Split reductive groups of semisimple rank 1.	414
g	Properties of SL_2	415
h	Classification of the split reductive groups of semisimple rank 1	418
i	The forms of SL_2 , GL_2 , and PGL_2	419
j	Classification of reductive groups of semisimple rank one	421
k	Review of SL_2	422
	Exercises	423
21	Split Reductive Groups	424
a	Split reductive groups and their roots	424
b	Centres of reductive groups	426
c	The root datum of a split reductive group	427
d	Borel subgroups; Weyl groups; Tits systems	432
e	Complements on semisimple groups	439
f	Complements on reductive groups	442
g	Unipotent subgroups normalized by T	444

Contents

xiii

h	The Bruhat decomposition	446
i	Parabolic subgroups	453
j	The root data of the classical semisimple groups	457
	Exercises	462
22	Representations of Reductive Groups	464
a	The semisimple representations of a split reductive group	464
b	Characters and Grothendieck groups	474
c	Semisimplicity in characteristic zero	476
d	Weyl's character formula	480
e	Relation to the representations of $\text{Lie}(G)$	482
	Exercises	483
23	The Isogeny and Existence Theorems	484
a	Isogenies of groups and of root data	484
b	Proof of the isogeny theorem	488
c	Complements	494
d	Pinnings	496
e	Automorphisms	499
f	Quasi-split forms	501
g	Statement of the existence theorem; applications	503
h	Proof of the existence theorem	505
i	The Langlands dual group	513
	Exercises	514
24	Construction of the Semisimple Groups	515
a	Deconstructing semisimple algebraic groups	515
b	Generalities on forms of semisimple groups	517
c	The centres of semisimple groups	519
d	Semisimple algebras	521
e	Algebras with involution	524
f	The geometrically almost-simple groups of type A	527
g	The geometrically almost-simple groups of type C	529
h	Clifford algebras	530
i	The spin groups	535
j	The geometrically almost-simple group of types B and D	536
k	The classical groups in terms of sesquilinear forms	538
l	The exceptional groups	541
m	The triality groups (groups of subtype 3D_4 and 6D_4)	545
	Exercises	545
25	Additional Topics	547
a	Parabolic subgroups of reductive groups	547
b	The small root system	551
c	The Satake–Tits classification	555

d	Representation theory	557
e	Pseudo-reductive groups	560
f	Nonreductive groups: Levi subgroups	562
g	Galois cohomology	563
	Exercises	568
Appendix A Review of Algebraic Geometry		569
a	Affine algebraic schemes	569
b	Algebraic schemes	572
c	Subschemes	574
d	Algebraic schemes as functors	576
e	Fibred products of algebraic schemes	578
f	Algebraic varieties	579
g	The dimension of an algebraic scheme	580
h	Tangent spaces; smooth points; regular points	581
i	Étale schemes over k	583
j	Galois descent for closed subschemes	584
k	Flat and smooth morphisms	585
l	The fibres of morphisms	586
m	Complete schemes; proper maps	587
n	The Picard group	588
o	Flat descent	588
Appendix B Existence of Quotients of Algebraic Groups		590
a	Equivalence relations	590
b	Existence of quotients in the finite affine case	595
c	Existence of quotients in the finite case	600
d	Existence of quotients in the presence of quasi-sections	603
e	Existence generically of a quotient	606
f	Existence of quotients of algebraic groups	608
Appendix C Root Data		611
a	Preliminaries	611
b	Reflection groups	612
c	Root systems	614
d	Root data	617
e	Duals of root data	619
f	Deconstructing root data	624
g	Classification of reduced root systems	624
	References	631
	Index	641