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ALGEBRAIC GROUPS

Algebraic groups play much the same role for algebraists as Lie groups play for analysts. This book is the first comprehensive introduction to the theory of algebraic group schemes over fields that includes the structure theory of semisimple algebraic groups and is written in the language of modern algebraic geometry.

The first eight chapters study general algebraic group schemes over a field and culminate in a proof of the Barsotti–Chevalley theorem realizing every algebraic group as an extension of an abelian variety by an affine group. After a review of the Tannakian philosophy, the author provides short accounts of Lie algebras and finite group schemes. The later chapters treat reductive algebraic groups over arbitrary fields, including the Borel–Chevalley structure theory. Solvable algebraic groups are studied in detail. Prerequisites have been kept to a minimum so that the book is accessible to non-specialists in algebraic geometry.

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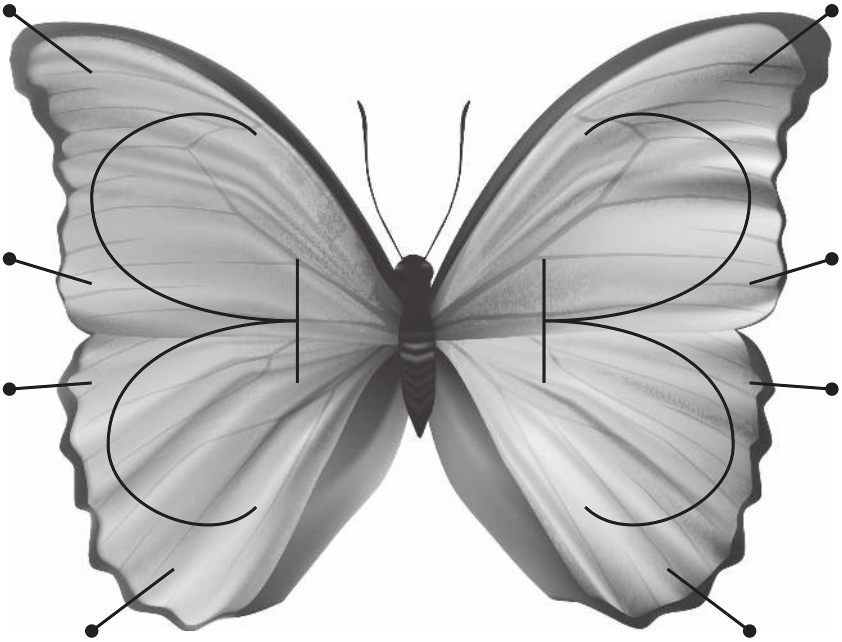

The picture illustrates Grothendieck's vision of a pinned reductive group: the body is a maximal torus T, the wings are the opposite Borel subgroups B, and the pins rigidify the situation. ("Demazure nous indique que, derrière cette terminologie [épinglage], il y a l'image du papillon (que lui a fournie Grothendieck): le corps est un tore maximal T, les ailes sont deux sous-groupes de Borel opposées par rapport à T, on déploie le papillon en étalant les ailes, puis on fixe des éléments dans les groupes additifs (des *épingles*) pour rigidifier la situation." SGA 3, XXIII, p. 177.)

The background image is a Blue Morpho butterfly. Credit: LPETTET/DigitalVision Vectors/Getty Images.

Algebraic Groups

The Theory of Group Schemes of Finite Type over a Field

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Preface

For one who attempts to unravel the story, the problems are as perplexing as a mass of hemp with a thousand loose ends.
Dream of the Red Chamber, Tsao Hsueh-Chin.

This book represents my attempt to write a modern successor to the three standard works, all titled *Linear Algebraic Groups*, by Borel, Humphreys, and Springer. More specifically, it is an exposition of the theory of group schemes of finite type over a field, based on modern algebraic geometry, but with minimal prerequisites.

It has been clear for fifty years that such a work has been needed.[1] When Borel, Chevalley, and others introduced algebraic geometry into the theory of algebraic groups, the foundations they used were those of the period (e.g., Weil 1946), and most subsequent writers on algebraic groups have followed them. Specifically, nilpotents are not allowed, and the terminology used conflicts with that of modern algebraic geometry. For example, algebraic groups are usually identified with their points in some large algebraically closed field K, and an algebraic group over a subfield k of K is an algebraic group over K equipped with a k-structure. The kernel of a k-homomorphism of algebraic k-groups is an object over K (not k) *which need not be defined over* k.

In the modern approach, nilpotents are allowed,[2] an algebraic k-group is intrinsically defined over k, and the kernel of a homomorphism of algebraic groups over k is (of course) defined over k. Instead of identifying an algebraic group with its points in some "universal" field, it is more convenient to identify it with the functor of k-algebras it defines.

The advantages of the modern approach are manifold. For example, the infinitesimal theory is built into it from the start instead of entering only in an ad hoc fashion through the Lie algebra. The Noether isomorphism theorems hold for

[1] "Another remorse concerns the language adopted for the algebrogeometrical foundation of the theory ... two such languages are briefly introduced ... the language of algebraic sets ... and the Grothendieck language of schemes. Later on, the preference is given to the language of algebraic sets ... If things were to be done again, I would probably rather choose the scheme viewpoint ... which is not only more general but also, in many respects, more satisfactory." Tits 1968, p. 2.

[2] To anyone who asked why we need to allow nilpotents, Grothendieck would say that they are already there in nature; neglecting them obscures our vision.

algebraic group schemes, and so the intuition from abstract group theory applies. The kernels of infinitesimal homomorphisms become visible as algebraic group schemes.

The first systematic exposition of the theory of group schemes was in SGA 3. As was natural for its authors (Demazure, Grothendieck, ...), they worked over an arbitrary base scheme and they used the full theory of schemes (EGA and SGA). Most subsequent authors on group schemes have followed them. The only books I know of that give an elementary treatment of group schemes are Waterhouse 1979 and Demazure and Gabriel 1970. In writing this book, I have relied heavily on both, but neither goes very far. For example, neither treats the structure theory of reductive groups, which is a central part of the theory.

As noted, the modern theory is more general than the old theory. The extra generality gives a richer and more attractive theory, but it does not come for free: some proofs are more difficult (because they prove stronger statements). In this work, I have avoided any appeal to advanced scheme theory. Unpleasantly technical arguments that I have not been able to avoid have been placed in separate sections where they can be ignored by all but the most serious students. By considering only schemes algebraic over a field, we avoid many of the technicalities that plague the general theory. Also, the theory over a field has many special features that do not generalize to arbitrary bases.

Acknowledgements: The exposition incorporates simplifications to the general theory from Iversen 1976, Luna 1999, Steinberg 1999, Springer 1998, and other sources. In writing this book, the following works have been especially useful to me: Demazure et al. 1966; Demazure and Gabriel 1970; Waterhouse 1979; the expository writings of Springer, especially Springer 1994, 1998; online notes of Casselman, Ngo, Perrin, and Pink, as well as the discussions, often anonymous, on https://mathoverflow.net/. Also I wish to thank all those who have commented on the various notes posted on my website.

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